Dcpo models of $T_1$ spaces

Zhao Dongsheng and Xi Xiaoyong

1 Mathematics and Mathematics Education
National Institute of Education Singapore
Nanyang Technological University
1 Nanyang Walk
Singapore 637616
dongsheng.zhao@nie.edu.sg

2 Department of Mathematics
Jiangsu Normal University
Jiangsu China
littlebrook@jsnu.edu.cn

A poset model of a topological space $X$ is a poset $P$ together with a homeomorphism $\phi : X \rightarrow \text{Max}(P)$ ($\text{Max}(P)$ is the subspace of the Scott space $\Sigma P$ consisting of maximal points of $P$). In [11] (also in [2]), it was proved that every $T_1$ space has a bounded complete algebraic poset model. It is, however, still unclear whether each $T_1$ space has a dcpo model. In this paper, we give a positive answer to this problem. In section 1, we show that every $T_1$ space has a dcpo model. In section 2, we prove that a $T_1$ space is sober if and only if its dcpo model constructed in section 1 is a sober dcpo. These results provide us with a method to construct non-sober dcpos from any non-sober $T_1$ spaces. In section 3, for some special spaces we construct a more concrete dcpo model.

1 Dcpo models of $T_1$ spaces

Theorem 1. [11] Every $T_1$ space has a bounded complete algebraic poset model.

Remark 1. Let $X$ be a $T_0$ space and $A$ be the set of all filters of open sets of $X$ that has a nonempty intersection. Then $(A, \subseteq)$ is a bounded complete algebraic poset and the following properties hold:

(1) the mapping $\phi : X \rightarrow \Sigma A$, defined by $\phi(x) = N(x), x \in X$ ($N(x)$ is the filter of open neighbourhood of $x$), is a topological embedding;

(2) $\text{Max}(A) \subseteq \phi(X)$, and $X$ is $T_1$ if and only if $\phi(X) = \text{Max}(A)$;

(3) every member of $A$ is below some $N(x)$, so the closure of $\phi(X)$ in $\Sigma A$ equals $A$.

Thus every $T_0$ space is homeomorphic to a dense subspace of the Scott space of a bounded complete algebraic poset.

A poset $P$ is called a local dcpo (or bounded complete dcpo) if every upper bounded directed subset has a supremum [12]. Clearly, every bounded complete poset is a local dcpo.

Lemma 1. For any local dcpo $A$, there is a dcpo $\hat{A}$ such that $\text{Max}(A)$ and $\text{Max}(\hat{A})$ are homeomorphic.

A poset $P$ is locally quasicontinuous if for each $a \in P$, the sub poset $\downarrow a$ is quasicontinuous.

Lemma 2. If $A$ is a bounded complete algebraic poset, then the dcpo $\hat{A}$ constructed in Lemma 1 from $A$ is locally quasicontinuous.
Given a $T_1$ space, by Theorem 1 there is a bounded complete algebraic poset $A$ such that $\text{Max}(A)$ is homeomorphic to $X$. Since every bounded complete poset is a local dcpo, by Lemma 3 there is a dcpo $\hat{A}$ such that $\text{Max}(A)$ is homeomorphic to $\text{Max}(A)$. All these deduce the first main result of this paper.

**Theorem 2.** Every $T_1$ topological space has a dcpo model.

**Remark 2.** By Lemma 2, we can actually deduce that every $T_1$ space has a dcpo model that is locally quasicontinuous.

**Proposition 1.** Every $T_0$ space can be embedded, as a dense subset, into the Scott space of an algebraic dcpo.

## 2 Dcpo models of sober spaces

**Proposition 2.** If $P$ is a poset such that $\Sigma P$ is sober, then the subspace $\text{Max}(P)$ of $\Sigma P$ is sober.

By Proposition III-3.7 of [3], the Scott space of every quasicontinuous dcpo is sober, so we have the following result.

**Corollary 1.** For any quasicontinuous dcpo, in particular for any continuous dcpo $P$, $\text{Max}(P)$ is sober.

**Lemma 3.** Let $A$ be a bounded complete algebraic poset and $\hat{A}$ be the dcpo constructed from $A$ in Lemma 1. If $\text{Max}(\hat{A})$ is sober then $\Sigma \hat{A}$ is sober.

From the above two results we deduce the following.

**Theorem 3.** A topological space $X$ has a dcpo model whose Scott topology is sober if and only if $X$ is $T_1$ and sober.

We call a dcpo $P$ sober, if its Scott topology is sober. Johnstone first constructed a non-sober dcpo in [5], then Isbel gave a non-sober complete lattice [4]. Finding a non-sober dcpo is surprisingly uneasy (as far as the authors know, up-to-date, only three such dcpos have been constructed).

Now if $X$ is a $T_1$ and non-sober space, then the dcpo model constructed for $X$ in Theorem 2 is non-sober.

For a specific example, let $Y$ be an infinite set and $\tau$ be the co-finite topology on $Y$ (i.e. $U \in \tau$ if and only if either $U = \emptyset$ or $Y - U$ is a finite set). Then $(Y, \tau)$ is $T_1$ and non-sober.

**Proposition 3.** Let $Q$ be a dcpo model of $(Y, \tau)$. Then $Q$ is a non-sober dcpo.

## 3 Dcpo models of some special spaces

Let $\omega_1$ be the first non-countable ordinal and $W = [0, \omega_1)$ be the set of all ordinals less than $\omega_1$. Thus $W$ consists of all finite and infinite countable ordinals.

**Remark 3.** The following facts are well known. 1) $|W| = \aleph_1$.

2) For any countable subset $D \subseteq W$, $\text{sup} D \in W$, here the $\text{sup} D$ is taken with respect to the usual linear order on ordinals.

3) For any $\alpha \in W$, $\{\beta : \beta \leq \alpha\}$ is a finite or countably infinite subset of $W$. 

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Let $\tau$ be the co-countable topology on $W$, that is $U \in \tau$ if and only if either $U = \emptyset$ or $W - U$ is a finite or countably infinite set. We now construct a simpler dcpo model for $(W, \tau)$.

Let $P_{\aleph_0} = \{x_\alpha : x \in W, \alpha \in W\} \cup W$. The order on $P_{\aleph_0}$ is defined as follows:

(i) $x_\alpha \leq y_\beta$ iff $\alpha = \beta$ and $x \leq y$;

(ii) $x_\alpha < \alpha$;

(iii) $x_\alpha < \beta$, where $\alpha \neq \beta$, iff $x < \beta$.

Then $P_{\aleph_0}$ is a dcpo and $\text{Max}(P_{\aleph_0}) = W$.

**Lemma 4.** (1) For any finite or countably infinite subset $A \subseteq W$, there is a Scott closed set $F$ of $P_{\aleph_0}$ such that $A = F \cap W$.

(2) For any Scott closed set $F$ of $P_{\aleph_0}$, either $W \subseteq F$ or $W - F$ is at most a countably infinite set.

**Proposition 4.** The dcpo $P_{\aleph_0}$ defined above is a model of the space of set $W = [0, \omega_1)$ with the co-countable topology.

As $W$ is not sober, its dcpo model $P_{\aleph_0}$ is non-sober in the Scott topology. This gives another example of non-sober dcpo.

In general, let $\aleph$ be a cardinal and $W_\aleph$ be the set of all ordinals $\alpha$ with $|\alpha| < \aleph$. The $\aleph$-complementary topology $\mu$ on $W_\aleph$ is the topology whose open sets are either $\emptyset$ or whose complement has cardinal less than or equal to $\aleph$. Then we can construct a dcpo model of $(W_\aleph, \mu)$ in a similar way as for $(W, \tau)$.

**Remark 4.** (1) Following the method as for Lemma 4, let $\mathbb{N}$ be the set of all natural numbers and $\tau$ the co-finite topology on $\mathbb{N}$. Let $P = \{n_k : n, k \in \mathbb{N}\} \cup \mathbb{N}$. Define the partial order $\leq$ on $P$ by

$m_k \leq n$ for any $k \leq n, n_k \leq m_l$ iff $m = n$ and $k \leq l$.

Then $P$ is a dcpo model of $(\mathbb{N}, \tau)$ where $\tau$ is the co-finite topology.

(2) In [5], Johnstone gives an example of a dcpo whose Scott topology is not sober (this is the first such example ever constructed). One can verify that this dcpo isomorphic to the dcpo $P$ defined in (1).

A dcpo model $P$ of a $T_1$ space $X$ is said to satisfy the Lawson condition if $X$ is homeomorphic to $\text{Max}(P)$ with the inherited Lawson topology on $P$. Lawson proved that a space has a continuous dcpo model satisfying Lawson condition that has a countable base iff the space is Polish[7]. In [11], it was proved that a space has an algebraic poset model satisfying Lawson condition iff it is zero-dimensional.

**Theorem 4.** If a space is zero dimensional then it has a dcpo model satisfying Lawson condition.

**References**


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