Strong 0-dimensionality in Pointfree Topology

Bernhard Banaschewski

McMaster University, Hamilton, Ontario, Canada
iscoe@math.mcmaster.ca

Abstract

Classically, a Tychonoff space is called strongly 0-dimensional if its Stone-Čech compactification is 0-dimensional, and given the familiar relationship between spaces and frames it is then natural to call a completely regular frame strongly 0-dimensional if its compact completely regular coreflection is 0-dimensional (meaning: is generated by its complemented elements). Indeed, it is then seen immediately that a Tychonoff space is strongly 0-dimensional iff the frame of its open sets is strongly 0-dimensional in the present sense.

This talk will provide an account of various aspects of this notion. Particuarly relevant for this will be

– the saturation quotient \( SM \) of a compact normal frame \( M \), given by the saturation nucleus \( s_M \) on \( M \) for which \( s_M(a) = \bigvee\{x \in M \mid x \lor y = e \Rightarrow a \lor y = e\} \).

– the variants of normality expressed by the following conditions: if \( a \lor b = e \) (the top) then there exists \( c \leq b \) such that \( a \lor c = e \) for which

\[
\begin{align*}
    c &< b \text{ (normal)} & c &< c^* \text{ (completely normal)} & c &\text{ complemented (strongly normal)}
\end{align*}
\]

where \( c < b \) means \( b \lor c^* = e \) (\( c^* \) the pseudocomplement of \( c \)) and \( c \ll b \) indicates the existence of an infinite sequence of interpolations

\[
    c < b, \; c < d_{11} < b, \; c < d_{21} < d_{11} < d_{23} < b, \ldots
\]

– the cozero part \( \text{Coz}L \) of a completely regular frame \( L \) consisting of all elements \( \text{coz}(\gamma) = \gamma((-,0) \lor (0,-)) \) for the real-valued continuous functions \( \gamma \) on \( L \) (corresponding to the classical cozero sets of a space), and

– a representation of the compact completely regular coreflection of a completely regular frame \( L \) as

\[
S_J \text{Coz}L \rightarrow L, \; I \mapsto \bigvee \{\text{coz}(\gamma) \mid \text{coz}(\gamma) \in I\}
\]

where \( J \text{Coz}L \) is the frame of ideals of the lattice \( \text{Coz}L \).

The latter will be used to obtain the following pointfree form of a classical result: A completely regular frame \( L \) is strongly 0-dimensional iff every cozero elements of \( L \) is a countable join of complemented ones.

Further, it will be shown that some familiar types of completely regular frames are strongly 0-dimensional and characterized by very natural additional conditions. This will include the \( P \)-frames, that is, the completely regular frames all whose cozero elements are complemented; they turn out to be the strongly 0-dimensional frames in which any countable join of complemented elements is complemented. The latter are called the \( P_0 \)-frames, and it will be shown that they are reflective in the category of all 0-dimensional frames. This parallels the corresponding result for \( P \)-frames in relation to all completely regular frames, but with a substantially simpler proof. It remains a challenging open question whether \( P_0 = P \).