Walker's Cancellation Theorem

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Abstract

Walker's cancellation theorem says that if $B \oplus \mathbf{Z}$ is isomorphic to $C \oplus \mathbf{Z}$ in the category of abelian groups, then B is isomorphic to C. We construct an example in a diagram category of abelian groups where the theorem fails. As a consequence, the original theorem does not have a constructive proof. In fact, in our example B and C are subgroups of $\mathbf{Z}^2$. Both of these results contrast with a group whose endomorphism ring has stable range one, which allows a constructive proof of cancellation and also a proof in any diagram category.

1 Cancellation

An object G in an additive category is **cancellable** if whenever $B \oplus G$ is isomorphic to $C \oplus G$, then B is isomorphic to C. Walker [6] and independently Cohn [3] answered a question of Kaplansky by showing that finitely generated abelian groups are cancellable in the category of abelian groups. The most interesting case is that of the integers $\mathbf{Z}$. That's because finitely generated groups are direct sums of copies of $\mathbf{Z}$ and of cyclic groups of prime power order, and a cyclic group of prime power order has a local endomorphism ring, hence is cancellable by a theorem of Azumaya [2].

It is somewhat anomalous that $\mathbf{Z}$ is cancellable. A rank-one torsion-free group A is cancellable if and only if $A \cong \mathbf{Z}$ or the endomorphism ring of A has stable range one [1, Theorem 8.12],[4]. (A ring R has **stable range one** if whenever $aR + bR = R$, then $a + bR$ contains a unit of R.) Thus for rank-one torsion-free groups, the endomorphism ring tells the whole story—except for $\mathbf{Z}$. It turns out that an object is cancellable if its endomorphism ring has stable range one. The proof of this in [5, Theorem 4.4] is constructive and works for any abelian category. It is also true, [5], that semilocal rings have stable range one, so Azumaya's theorem is a special case of this. In fact, that the endomorphism ring of A has stable range one is equivalent to A being substitutable, a stronger condition than cancellation [5, Theorem 4.4]. We say that A is **substitutable** if any two summands of a group, with complements that are isomorphic to A, have a common complement. The group $\mathbf{Z}$ is not substitutable: Consider the subgroups of $\mathbf{Z}^2$ generated by $(1,0)$, $(0,1)$, $(7,3)$, and $(5,2)$. The first and second, and the third and fourth, are complementary summands. The second and fourth do not have a common complement because that would require (a,b) with $a = \pm 1$ and $2a - 5b = \pm 1$.

In this paper we will investigate whether $\mathbf{Z}$ is cancellable in the (abelian) category $\mathcal{D}_T(\mathbf{Ab})$ of diagrams of abelian groups based on a fixed finite poset T with a least element. There is a natural embedding of $\mathbf{Ab}$ into $\mathcal{D}_T(\mathbf{Ab})$ given by taking a group into the constant diagram on T with identity maps between the groups on the nodes. In particular, we can identify the group of integers as an object of $\mathcal{D}_T(\mathbf{Ab})$. As the endomorphism ring of any group G is the same as that of its avatar in $\mathcal{D}_T(\mathbf{Ab})$, a substitutable group is substitutable viewed as an object in $\mathcal{D}_T(\mathbf{Ab})$. However it turns out that $\mathbf{Z}$ is not cancellable in $\mathcal{D}_T(\mathbf{Ab})$ where T is the linearly ordered set $\{0, 1, 2\}$.

It follows that Walker's theorem does not admit a constructive proof, because the construction can be viewed as a Kripke model counter-example. In fact, it is not even provable for B and C subgroups of $\mathbf{Z}^2$. It was the question of whether Walker's theorem had a constructive proof that initiated our investigation. (Most any proof of Azumaya's theorem is constructive.)

As any homomorphism from an abelian group onto $\mathbf{Z}$ splits, Walker's theorem can be phrased as follows: If A is an abelian group, and $f, g : A \to \mathbf{Z}$ are epimorphisms, then $\ker f \cong \ker g$. The following theorem gets us part way to a proof of Walker's theorem.

Theorem 1. *Let A be an abelian group and $f, g : A \to \mathbf{Z}$ be epimorphisms. Then $f(\ker g) = g(\ker f)$ so that*

$$\frac{\ker g}{\ker f \cap \ker g} \cong f(\ker g) = g(\ker f) \cong \frac{\ker f}{\ker f \cap \ker g}$$

Thus we get the desired isomorphism $\ker f \cong \ker g$ if $\ker f \cap \ker g = 0$ or if $f(\ker g)$ is projective. Classically, every subgroup of $\mathbf{Z}$ is projective, so this constitutes a classical proof.

2 The example

Our example lives in the category $\mathcal{D}_T(\mathbf{Ab})$ of diagrams of abelian groups based on the linearly ordered set $T = \{0, 1, 2\}$. The example shows that you can't cancel $\mathbf{Z}$ in $\mathcal{D}_T(\mathbf{Ab})$.

The groups on the nodes will be subgroups $A_0 \subset A_1 \subset A_2 = \mathbf{Z}^3$ defined by generators:

$$A_0 = \begin{matrix}(1,3,0)\\(3,1,0)\end{matrix} \quad A_1 = \begin{matrix}(1,0,-24)\\(0,1,8)\\(0,0,64)\end{matrix} \quad A_2 = \begin{matrix}(1,0,0)\\(0,1,0)\\(0,0,1)\end{matrix}$$

Note that $(0,8,0), (8,0,0) \in A_0$. The maps between these groups are inclusions. Define the maps $f, g : \mathbf{Z}^3 \to \mathbf{Z}$ by $f(a,b,c) = a$ and $g(a,b,c) = b$. The maps f and g each induce maps from these three groups into $\mathbf{Z}$ which give two maps from the diagram into the constant diagram $\mathbf{Z}$. We denote the kernel of the map f restricted to A_i by $\ker_i f$ and similarly for g. These kernels admit the following generators:

$$\ker_0 f = (0,8,0) \quad \ker_1 f = \begin{matrix}(0,1,8)\\(0,0,64)\end{matrix} \quad \ker_2 f = \begin{matrix}(0,1,0)\\(0,0,1)\end{matrix}$$

$$\ker_0 g = (8,0,0) \quad \ker_1 g = \begin{matrix}(1,0,-24)\\(0,0,64)\end{matrix} \quad \ker_2 g = \begin{matrix}(1,0,0)\\(0,0,1)\end{matrix}$$

The diagrams $B = \ker f$ and $C = \ker g$ are clearly each embeddable in the diagram $\mathbf{Z} \oplus \mathbf{Z}$. That $B \oplus \mathbf{Z}$ is isomorphic to $C \oplus \mathbf{Z}$ follows from the fact that the diagram A can be written as an internal direct sum $B \oplus \mathbf{Z}$ and also as an internal direct sum $C \oplus \mathbf{Z}$. The generator of $\mathbf{Z}$ in the first case is the element $(1,3,0)$, in the second case $(3,1,0)$.

Theorem 2. *There is no isomorphism between* $\ker f$ *and* $\ker g$ *in* $\mathcal{D}_T(\mathbf{Ab})$.

The following result shows that we can't get an example that is a subobject of the diagram $\mathbf{Z}^n$ using the linearly ordered set $T = \{0, 1\}$.

Theorem 3. *Let $T = \{0, 1\}$. In the category $\mathcal{D}_T(\mathbf{Ab})$, if A and B are subobjects of $\mathbf{Z}^n$, and $A \oplus \mathbf{Z}$ is isomorphic to $B \oplus \mathbf{Z}$, then A is isomorphic to B.*

This theorem leaves open the question of whether there is an counterexample of this sort using the poset that looks like a "V".

3 Canceling Z with respect to subgroups of Q

So we can't cancel $\mathbf{Z}$ with respect to certain subgroups of $\mathbf{Z} \oplus \mathbf{Z}$. What is the situation is with respect to subgroups of $\mathbf{Z}$? Constructively:

Theorem 4. *Let B be an abelian group such that every nontrivial homomorphism from B to $\mathbf{Z}$ is one-to-one. If f is a homomorphism from $B \oplus \mathbf{Z}$ onto $\mathbf{Z}$, then $\ker f$ is isomorphic to mB for some positive integer m. Hence if B is torsion free, then $\ker f$ is isomorphic to B.*

Note that any torsion-free group B of rank at most one satisfies the hypothesis of the theorem.

What other groups B allow cancellation of $\mathbf{Z}$? It suffices that B be finitely generated. To see this, look at Theorem 1. If $\ker f$ is finitely generated, then $g(\ker f)$ is a finitely generated subgroup of $\mathbf{Z}$, hence is projective. From this argument it suffices that any image of B in $\mathbf{Z}$ be finitely generated. Notice that subgroups of $\mathbf{Z}$ need not have this property. What about a direct sum of two groups that allow cancellation of $\mathbf{Z}$, such as a direct sum of two subgroups of $\mathbf{Z}$?

References

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