Extensions of ordering sets of states from effect algebras onto their MacNeille completions

Jiří Janda[1] and Zdenka Riečanová[2]

[1] Department of Mathematics and Statistics, Faculty of Science, Masaryk University, Kotlářská 2, CZ-611 37 Brno, Czech Republic.
98599@mail.muni.cz

[2] Department of Mathematics, Faculty of Electrical Engineering and Information Technology, Slovak University of Technology, Ilkovičova 3, SK-812 19 Bratislava, Slovak Republic.
zdenka.riecanova@stuba.sk

Keywords: effect algebra, MV-effect algebra, MacNeille completion, positive linear operators in Hilbert space, Hilbert space effect-representation.

Mathematics Subject Classification (2000): 06F05, 03G25, 81P10, 08A55.

1 Introduction

The notion of an effect algebra was presented by Foulis and Bennett in (Foulis, Bennett, [3] 1994). The definition was motivated by giving an algebraic description of a logic of quantum effects $\mathcal{E}(\mathcal{H})$, i.e. the set of all positive self-adjoint operators between zero and identity operator I in a separable complex Hilbert space $\mathcal{H}$. On $\mathcal{E}(\mathcal{H})$ was defined a partial operation $A \oplus B = A + B$ iff $A + B \leq I$ with meaning of an orthogonal disjunction. Quantum effects in studies of quantum mechanics correspond to yes-no measurements that may be unsharp. An equivalent structure called *D-poset* has been introduced by Kôpka and Chovanec ([7] 1992, [6] 1994).

Definition 1 ([3])**.** A partial algebra $(E; \oplus, 0, 1)$ is called an *effect algebra* if $0, 1 \in E$ are two distinguished elements and $\oplus$ is a partially defined binary operation on E which satisfy the following conditions for any $x, y, z \in E$:

(Ei) $x \oplus y = y \oplus x$ if $x \oplus y$ is defined,

(Eii) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$ if one side is defined,

(Eiii) for every $x \in E$ there exists a unique $y \in E$ such that $x \oplus y = 1$ (we put $x' = y$),

(Eiv) if $1 \oplus x$ is defined then $x = 0$.

In every effect algebra E a relation $\leq$ can be defined by

(PO) $x \leq y$ iff $x \oplus z$ is defined and $x \oplus z = y$ for any $x, y, z \in E$.

Then $\leq$ is a partial order on E.

Definition 2. Let $(E; \oplus, 0_E, 1_E)$ be an effect algebra and $\omega : E \to [0, 1] \subseteq \mathbb{R}$ be a map such that $\omega(1_E) = 1$ and $\omega(x \oplus y) = \omega(x) + \omega(y)$ whenever $x \oplus y$ is defined. Then we call ω a *state* on E. A set $\mathcal{M}$ of states on E is called an *ordering set of states* if for any $x, y \in E$ condition $x \leq y$ iff $\omega(x) \leq \omega(y)$ for all $\omega \in \mathcal{M}$ is satisfied.

Assume that $(E;\oplus,0,1)$ is an effect algebra possessing an ordering set $\mathcal{M}=\{\omega : E\to[0,1] \mid \omega \text{ is a state on } E\}$ of states on E. Let $l_2(\mathcal{M})=\{(x_\omega)_{\omega\in\mathcal{M}} \mid x_\omega\in\mathbb{C}, \sum_{\omega\in\mathcal{M}}|x_\omega|^2<\infty\}$ be a complex Hilbert space with the usual inner product $\langle (x_\omega)_{\omega\in\mathcal{M}},(y_\omega)_{\omega\in\mathcal{M}}\rangle=\sum_{\omega\in\mathcal{M}}\overline{x}_\omega\cdot y_\omega$.

In [9] it was proved that: *Every effect algebra* $(E;\oplus,0,1)$ *with ordering set* $\mathcal{M}$ *of states on* E *can be EA-embedded into the Hilbert space effect algebra* $\mathcal{E}(l_2(\mathcal{M}))=[\mathbf{0},I]_{\mathcal{B}^+(l_2(\mathcal{M}))}$. We call this embedding a *Hilbert space effect-representation* of E and E is called *Hilbert space effect-representable.*

It is well known that for every poset $(P,\le)$ there exists a Dedekind-MacNeille completion $MC(P)$, that is a complete lattice in which can P be order densely embedded. For the case of effect algebras, as partially ordered sets with respect to induced partial order, it may happened that for an effect algebra E a partial sum $\hat{\oplus}$ cannot be defined in $MC(E)$ in the way that its restriction on E coincide with partial sum $\oplus$ on E (see [8]). Whenever such partial operation $\hat{\oplus}$ exists on $MC(E)$, we call $\hat{E}=(MC(E);\hat{\oplus},0,1)$ an *effect algebraic MacNeille completion* (shortly *EA-MacNeille completion*).

We consider the problem for an effect algebra E which has an effect algebraic MacNeille completion $\hat{E}$ and has a Hilbert space representation in $\mathcal{E}(l_2(\mathcal{M}))$ as well, in which cases we can represent $\hat{E}$ in the same Hilbert space operator effect algebra $\mathcal{E}(l_2(\mathcal{M}))$. That is when the ordering set $\mathcal{M}$ of states on E can be be extended to an ordering set $\hat{\mathcal{M}}$ of states on $\hat{E}$, hence $|\mathcal{M}|=\left|\hat{\mathcal{M}}\right|$ and $\mathcal{E}(l_2(\mathcal{M}))=\mathcal{E}(l_2(\hat{\mathcal{M}}))$.

2 Hilbert space effect-representation of an effect algebra and its EA-MacNeille completion

The following theorem states, that if extension of the system of the states on EA-MacNeille completion exists, than it preserves the ordering property.

Theorem 1. [5] *Assume that* $(E;\oplus,0,1)$ *is an effect algebra possessing an ordering set* $\mathcal{M}$ *of states on* E*. Further let* E *has an EA-MacNeille completion* $\hat{E}=(MC(E),\hat{\oplus},0,1)$ *and let* E *be identified with* $\varphi(E)$*, where* $\varphi: E\to\hat{E}$ *is supremum and infimum dense effect algebraic embedding of* E *into* $\hat{E}$*. Let for every state* $\omega\in\mathcal{M}$ *there exists an extension to a state* $\hat{\omega}$ *on* $\hat{E}$*.*

Then $\hat{\mathcal{M}}=\{\hat{\omega}\mid \hat{\omega} \text{ is a state on } \hat{E}, \hat{\omega}_{|E}=\omega\in\mathcal{M}\}$ *is an ordering set of states on* $\hat{E}$*.*

Corollary 1. [5] *Under the assumptions of Theorem 1 on effect algebra* $(E;\oplus,0,1)$ *and its EA-MacNeille completion* $\hat{E}=(MC(E),\hat{\oplus},0,1)$*, the following conditions are satisfied:*

(i) *Effect algebras* E *and* $\hat{E}$ *are Hilbert space effect-representable.*

(ii) $l_2(\mathcal{M})=l_2(\hat{\mathcal{M}})$ *and* E *and its EA-MacNeille completion* $\hat{E}$ *can be both embedded into* $\mathcal{E}(l_2(\hat{\mathcal{M}}))=[0,I]_{l_2(\hat{\mathcal{M}})}$.

(iii) $\varphi(\hat{E})$ *is an EA-MacNeille completion of* $\varphi(E)$ *where* $\varphi:\hat{E}\to\mathcal{E}(l_2(\hat{\mathcal{M}}))$ *is the EA-embedding in (ii) and* $\varphi(E)\subseteq\varphi(\hat{E})$*.*

Elements of an effect algebra $(E;\oplus,0,1)$ are called *compatible* (we write $a\leftrightarrow b$) if there exists $a_1,c,b_1\in E$ such that $a_1\oplus c\oplus b_1$ is defined in E and $a=a_1\oplus c$, $b=b_1\oplus c$. A lattice effect algebra E with $a\leftrightarrow b$ for all $a,b\in E$ is called an *MV-effect algebra* and can be organized into an MV-algebra.

Theorem 2. [5] *Let* E *be an Archimedean MV-effect algebra and* $\mathcal{M}$ *be an ordering set of (o)-continuous states on* E*. Then*

(i) *Every state $\omega \in \mathcal{M}$ can be extended to a state $\hat{\omega}$ on the EA-MacNeille completion $\hat{E}$ of E.*

(ii) $\hat{\mathcal{M}} = \{\hat{\omega} \mid \hat{\omega}$ *is a state on* $\hat{E}$, $\hat{\omega}_{|E} = \omega \in \mathcal{M}\}$ *is an ordering set of states on* $\hat{E}$.

Corollary 2. [5] *Every Archimedean MV-effect algebra E and its EA-MacNeille completion $\hat{E}$ have Hilbert space effect-representations in the same Hilbert space $l_2(\hat{\mathcal{M}})$ where $\hat{\mathcal{M}}$ is an ordering set of states on $\hat{E}$ extending states form E.*

2.1 Atomic lattice effect algebras

Theorem 3. [5] *Let $(E; \oplus, 0, 1)$ be an (o)-continuous Archimedean atomic lattice effect algebra with an ordering set $\mathcal{M}$ of (o)-continuous states. Let E has an EA-MacNeille completion $\hat{E}$. Then*

(i) *To every $\omega \in \mathcal{M}$ there exists a unique (o)-continuous state $\hat{\omega}$ on $\hat{E}$ such that $\hat{\omega}_{|E} = \omega$.*

(ii) $\hat{\mathcal{M}} = \{\hat{\omega} \mid \hat{\omega}$ *is a state on* $\hat{E}$, $\hat{\omega}_{|E} = \omega \in \mathcal{M}\}$ *is an ordering set of states on* $\hat{E}$.

(iii) *Both effect algebras E and $\hat{E}$ have the Hilbert space effect-representations in $l_2(\mathcal{M}) = l_2(\hat{\mathcal{M}})$.*

Theorem 4. [5] *An Archimedean atomic distributive effect algebra $(E; \oplus, 0, 1)$ has a Hilbert space effect-representation if and only if $(E; \oplus, 0, 1)$ is an MV-effect algebra if and only if $(E; \oplus, 0, 1)$ is isomorphic to a sub-direct product of finite chains.*

A lattice effect algebra $(E; \oplus, 0, 1)$ is called *modular* if and only if E as a poset is a modular lattice.

Theorem 5. [5] *An Archimedean atomic modular lattice effect algebra $(E; \oplus, 0, 1)$ which is isomorphic to a sub-direct product of finite chains and modular diamonds as well as its MacNeille completion are Hilbert space effect-representable effect algebras.*

Acknowledgement 1. *Jiří Janda kindly acknowledges the support by Masaryk University, grant 0964/2009 and ESF Project CZ.1.07 /2.3.00/20.0051 Algebraic Methods in Quantum Logic of the Masaryk University. Zdenka Riečanová kindly acknowledges the support by the Science and Technology Assistance Agency under the contract APVV-0178-11 Bratislava SR, and VEGA-grant of MŠ SR No. 1/0297/11.*

References

[1] Blank J., Exner P., Havlíček M., *Hilbert Space Operators in Quantum Physics*, 2nd edn. Springer, Berlin 2008.

[2] Dvurečenskij A., Pulmannová S., *New Trends in Quantum Structures*, Kluwer Acad. Publ., Dordrecht/Ister Science, Bratislava, 2000.

[3] Foulis D. J., Bennett M. K., *Effect Algebras and Unsharp Quantum Logics*, *Found. Phys.*, **24** (1994), 1331–1352.

[4] Gudder S., *D-algebras*, Found. Phys. **26**, no. 6, (1996), 813-822.

[5] Janda, J., Riečanová Z., *Extensions of ordering sets of states from effect algebras onto their MacNeille completions* Inter. J.Theor. Phys., DOI 10.1007/s10773-013-1532-4.

[6] Kôpka F., Chovanec F. *D-posets*, Math. Slovaca **44**, 21, (1994).

[7] Kôpka F. *D-posets of Fuzzy Sets*, Tatra Mt. Math. Publ. **1**, (1992), pp. 83-87.

[8] Riečanová Z.,*MacNeille Completions of D-posets and Effect Algebras*, Inter. J. Theor. Phys. **39**, (2000), No.3., pp. 859–869.

[9] Riečanová Z, Zajac M., *Hilbert Space Effect-Representations of Effect Algebras*, Rep. Math. Phys. **70**, (2012), No. 2, pp. 283–290.