The Weak Convergence in Hilbert Spaces Concept Construction (Revisited)

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Abstract: We intend in this article critically review the weak convergence concept in the real Hilbert[1] spaces domain. Indeed, the purpose is to review, correct and enlarge the paper by Ferreira (2014) on this subject, where is studied mainly its construction process. The main goal is to enlarge the Bolzano[2] -Weierstrass[3] theorem field of validity. Then it is discussed in which conditions weak convergence implies convergence.

Keywords: Hilbert spaces; Bolzano-Weierstrass theorem; weak convergence.

***Mathematics Subject Classification*:** 46C15

Introduction

Begin with some notions on Hilbert spaces essential for the progress of this work:

Definition 1.1

A Hilbert space is a complex vector space with inner product that, as metric space, is complete.■

Usually we designate a Hilbert space H or *I*.

Definition 1.2

An inner product in a complex vector space H is a sesquilinear hermitian and strictly positive functional on H.■

Observation:

-In real vector spaces, "sesquilinear hermitian" must be changed to"bilinear symmetric",

-The inner product of two vectors x and y belonging to H, in this order, is denoted$[x, y]$,

-The norm of a vector x is given by $\|x\| = \sqrt{[x, x]}$,

-The distance between two vectors x and y belonging to H is $d(x, y) = \|x - y\|$.

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[1] **David Hilbert**, (January 23, 1862 – February 14, 1943) was a German mathematician, recognized as one of the most influential and universal mathematicians of the 19th and early 20th centuries. Hilbert discovered and developed a broad range of fundamental ideas in many areas, including invariant theory and the axiomatization of geometry. He also formulated the theory of Hilbert spaces one of the foundations of functional analysis.

[2] **Bernhard Placidus Johann Nepomuk Bolzano** (October 5, 1781 – December 18, 1848) was a Bohemian mathematician, logician, philosopher, theologian, Catholic priest and antimilitarist of German mother tong.

[3] **Karl Theodor Wilhelm Weierstrass** (31 October 1815 – 19 February 1897) was a German Mathematician who is often cited as the "father of modern analysis".

Proposition 1.1

The norm, in a space with inner product, satisfies the parallelogram rule.

$$\|x-y\|^2+\|x+y\|^2=2(\|x\|^2+\|y\|^2).\ ■$$

Now follows the presentation of the problem we wish to study.

Bolzano-Weierstrass theorem, establishes that a bounded sequence of real numbers has at least one sublimit. This result remains true for any finite dimension space with inner product.

Such a result does not stand when we consider infinite dimension spaces. Really, under those conditions it is possible to find a sequence in a Hilbert space *H*, orthonormal, designated $\{h_n\}$.So $\|h_n\|=1$ and

$$\|h_n-h_m\|^2=[h_n-h_m,h_n-h_m]=\|h_n\|^2+\|h_m\|^2=1+1=2, \text{if } m\neq n.$$

Consequently, this is a bounded sequence and has no sublimits. So, it is legitimate to ask which the generalization of the Bolzano-Weierstrass theorem is.

The problem of weak convergence in Hilbert Spaces is an important concept in theoretical mathematics, see (Ferreira *et al*, 2012), with multiple and interesting applications in real world problems.

A conceivable application of this notion is in the field of nonlinear time series. There stationarity and convergence are related concepts. In linear problems, weak stationarity plays a very important role. It is, however, uncommon to assume a truly nonlinear framework and the current article presents a thoughtful contribution to ease such applications. On this subject see, for instance, (Campbell *et al*, 1977), (Nigmatullin *et al*, 2013) and (Bentes and Menezes, 2013).

It is therefore an interesting and important contribution to the knowledge of the functionals mathematical properties defined in non-trivial complexity spaces; see (Royden, 1968) and (Kantorovich and Akilov, 1982).

This paper is the enlarged version of Ferreira (2014) on this subject.

Weak Convergence

For any $g\in H$ and for the orthonormal sequence seen above: $\|g\|^2\geq\sum_{k=1}^{\infty}|[g,h_k]|^2$, according to Bessel's inequality. So

$$\lim_k[g,h_k]=0=[g,0],\forall g\in H.$$

Originated on this example, we introduce a weaker notion of convergence.

Definition 2.1

A sequence x_k in *H* weakly converges for *x* belonging to *H* if and only if $\lim_k[x_k,g]=[x,g]$ for any *g* in *H*. ■

Definition 2.2

A *y* is a weak limit of a set *M* if and only if $[x,y]$ is a limit point of $[x,M]$ for any *x* in *H*. ■

Definition 2.3

A set *M* is weakly closed if and only if contains all its weak limits. ■

Observation:

-Every set weakly closed is closed. The reciprocal proposition is not true.

Now we will enounce, without demonstration, two theorems that establish important properties for Hilbert spaces. The second is true in any Banach[4] space. To demonstrate the first it would be necessary the Riez[5] representation theorem, see Ferreira (2013) and (Ferreira and Andrade, 2011), which enouncement and demonstration follow:

Theorem 2.1 (Riesz representation)

Every continuous linear functional $f(\cdot)$ may be represented in the form $f(x) = [x, \tilde{q}]$ where

$$\tilde{q} = \frac{\overline{f(q)}^{6}}{[q,q]} q.$$

Dem:

Begin noting that for every continuous linear functional $f(.)$, the *Nucleus* of $f(.)$ [7] is a closed vector subspace. If the functional under consideration is not the null functional, there is an element y such that $f(y) \neq 0$. Be z the projection of y over *Nuc* (*f*) and make $q = y - z$. So, q is orthogonal to *Nuc* (*f*), $f(q) = f(y)$ and, in consequence, $f(q) \neq 0$.

Then, for every $x \in H, x - \frac{f(x)}{f(q)} q$ belongs, evidently to *Nuc*(*f*). So, $x - \frac{f(x)}{f(q)} q$ is orthogonal to q and, in consequence, $[x,q] - \frac{f(x)}{f(q)}[q,q] = 0 \Leftrightarrow [x,q] = \frac{f(x)}{f(q)}[q,q]\ that\ is: f(x) = \left[x, \frac{\overline{f(q)}}{[q,q]} q\right]$. ■

Observation:

-From the theorem it results $\|f\|_{H'} = \|\tilde{q}\|_H$, where H' is the H dual space[8].

For the second it would be necessary the Baire[9] category theorem, see (Royden, 1968), true for any complete metric space.

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[4] **Stefan Banach** (March 30, 1892 – August 31, 1945) was a Polish matematician. He is generally considered to have been one of the 20th century's most important and influential mathematicians. Banach was one of the founders of modern functional analysis and one of the original members of the Lwów School of Mathematics. His major work was the 1932 book, *Théorie des opérations linéaires* (Theory of Linear Operations), the first monograph on the general theory of functional analysis.

[5] **Frigyes Riesz** (January 22, 1880 – February 28, 1956) was a Hungarian mathematician who made fundamental contributions to functional analysis.

[6] $\overline{f(q)}$ is the conjugate complex number of *f(q).*

[7] The *Nucleus* of $f(.)$is designated *Nuc*(*f*) and *Nuc*(*f*)={x: $f(x) = 0$}.

[8] Consider a continuous linear functional *f* in a normed space *E*. It is called *f* norm, and designated $\|f\|$:

$$\|f\| = \sup_{\|x\| \leq 1} |f(x)|$$

that is: the supreme of the values assumed by $|f(x)|$ in the *E* unitary ball. The class of the continuous linear functionals, with the norm above defined, is a normed vector space, called the *E* dual space, designated E'. Of course, a Hilbert space is a normed space.

[9] **René-Louis Baire** (21 January 1874 – 5 July 1932) was a French mathematician most famous for his Baire category theorem, which helped to generalize and prove future theorems. His theory was published originally in his dissertation *Sur les fonctions de variable réelles* ("On the Functions of Real Variables") in 1899.

Theorem 2.2 (Weak Compactness Property)

Every bounded sequence in a Hilbert space contains at least a subsequence weakly convergent. ■

Theorem 2.3 (Uniform Boundary Principle**)**

Be $f_n(.)$ a sequence of continuous linear functionals in *H* such that $\sup_n |f_n(x)| < \infty$ for each *x* in *H*. Then $\|f_n(.)\| \leq M$ for any $M < \infty$.■

Two very useful corollaries, from this theorem are:

Corollary 2.1

Be $f_n(.)$ a sequence of continuous linear functionals such that, for each $x \in H, f_n(x)$ converges. Then there is a continuous linear functional such that $f(x) = lim\, f_n(x)$ and $\|f(.)\| \leq \underline{lim}\, \|f_n(.)\|$.

Dem:

By the Uniform Boundary Principle, it follows that$\|f_n(.)\| \leq M$ for any $M < \infty$. Define $g(x) = lim f_n(x)$. So $g(.)$ is evidently linear. Suppose that $\|x_m - x\| \to 0$. So $|g(x_m - x)| = \lim_n |f_n(x_m - x)| \leq M\|x_m - x\| \to 0$.

Consequently, $g(.)$ is continuous. Also for any *x*,

$$\|x\| = 1, |g(x)| = lim|f_n(x)| \leq \underline{lim}\, \|f_n(.)\|.$$■

Corollary 2.2

Be $f_n(.)$ a sequence of continuous linear functional such that $\|f_n(.)\| \leq M$ and $f_n(.)$ converges for each *x* in a dense subset of *H*. Then,

-There is a linear continuous functional $f(.)$ such that $\lim_n f_n(x) = f(x)$ since this limit exists,

-The limit linear functional is unique.

Dem:

We will state that $f_n(x)$, indeed, converges for every *x* in *H*. For it, be x_n in the dense set[10] :

$$\|x - x_n\| \to 0;\ f_m(x_n) \text{ converges in } m.$$

Consider *p*, great enough such that, given $\varepsilon > 0, \|x - x_p\| \leq \frac{\varepsilon}{4M}$.

And *n* and *m* so that $|f_n(x_p) - f_m(x_p)| \leq \frac{\varepsilon}{2}$. So,

$$|f_m(x) - f_n(x)| \leq |f_m(x - x_p) - f_n(x - x_p)| + |f_m(x_p) - f_n(x_p)| \leq 2M\|x - x_p\| + \frac{\varepsilon}{2} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Then, $f_n(x)$ converges and the conditions of the former corollary are satisfied. ■

Weak Convergence and Convergence

It is obvious to pose the following question:

[10] That is: be x_n, elements of the dense set, such that $x_n \to x$.

-*Under which conditions weak convergence implies convergence*.

The first result important to answer this question is:

Theorem 3.1
Suppose that x_n converges weakly for *x* and $\|x_n\|$ for $\|x\|$. Then x_n converges for *x*.

Dem:
It is immediate that

$$\|x_n - x\|^2 = \|x_n\|^2 + \|x\|^2 - [x_n, x] - [x, x_n] \to \|x\|^2 + \|x\|^2 - 2[x, x] = 2\|x\|^2 - 2\|x\|^2 = 0.$$

Consequently, $\|x_n - x\|^2 \to 0$.■

Much more useful than the former one in the applications, on weak convergence, is the following result due to Banach-Saks[11]:

Theorem 3.2 (Banach-Saks)
Suppose that x_n converges weakly for *x*. Then it is possible to determine a subsequence $\{x_{n_k}\}$ such that the arithmetical means $\frac{1}{m}\sum_{k=1}^{m} x_{n_k}$ converge for *x*.

Dem:
Generality lossless, it may be supposed that *x* = 0. Consider x_{n_k} as follows:
-$x_{n_1} = x_1$,

-Due to the weak convergence, it is possible to choose x_{n_2}, such that $|[x_{n_1}, x_{n_2}]| < 1$,

-Having considered x_{n_1}, …, x_{n_k} it is evident that it is admissible to choose $x_{n_{k+1}}$ such that $|[x_{n_i}, x_{n_{k+1}}]| < \frac{1}{k}, i = 1, 2, \dots, k$.

As, by the uniform boundary, it is possible to take $\|x_{n_k}\| \leq M \text{ for any } M < \infty$, with the inner products usual calculations rules it is obtained:

$$\left\|\frac{1}{k}\sum_{i=1}^{k} x_{n_i}\right\|^2 \leq \left(\frac{1}{k}\right)^2 \left(kM + 2\sum_{i=2}^{k}\sum_{j=1}^{i-1}\left|[x_{n_j}, x_{n_i}]\right|\right) \leq \frac{1}{k^2}\left(kM + 2(k-1)\right) \to 0.$$

So $\frac{1}{m}\sum_{k=1}^{m} x_{n_k}$ converges to 0. ■

Observation:
-An alternative formulation of Theorem 3.2 is:

Every closed convex subset is weakly closed.

Finally, we present a Corollary of Theorem 3.2.

[11] **Stanisław Saks** (December 30, 1897 – November 23, 1942) was a Polish mathematician and university tutor, known primarily for his membership in the Scottish Café circle, an extensive monograph on the theory of integrals, his works on measure theory and the Vitali-Hahn-Saks theorem.

Corollary 3.1 (Convex Functionals Weak Inferior Semicontinuity)

Be $f(.)$ a continuous convex functional in the Hilbert space *H*. So if x_n weakly converges to *x*, $\underline{lim}\, f(x_n) \geq f(x)$.

Dem:

Consider a subsequence x_{n_m}, and put $x_m = x_{n_m}$, in order that $\underline{lim}\, f(x_n) = \lim f(x_m)$ and, still, that $\frac{1}{n}\sum_{m=1}^{n} x_m$ converges for *x*, in accordance with Theorem 3.2. However, as $f(.)$ is convex,

$$\frac{1}{n}\sum_{k=1}^{n} f(x_k) \geq f\left(\frac{1}{n}\sum_{k=1}^{n} x_k\right).$$

So, $\underline{lim}\, f(x_n) = \lim \frac{1}{n}\sum_{k=1}^{n} f(x_k) \geq \lim f\left(\frac{1}{n}\sum_{k=1}^{n} x_k\right) = f\,(x)$.■

Conclusions

The notion of weak convergence established in Definition 2.1 allows a possible Bolzano-Weierstrass theorem generalization. The Theorem 2.1 (Weak Compactness Property) and the Theorem 2.2 (Uniform Boundary Principle) help to understand that notion. Also in the Corollary 2.1 and in the Corollary 2.2 some operational properties are established. Finally, with the help of Banach-Saks Theorem we present conditions under which weak convergence implies convergence.

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