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A NOVEL BOUNDARY AVERAGING OPERATOR WITH AN APPLICATION FOR BEAM STRUCTURES

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Abstract. The paper presents a new boundary averaging operator (BAO) in which the particular role of boundary values is considered in a more suitable way than the conventional approach. A remarkable feature of BAO is that this operator contains a parameter of boundary regulation *p* and depends on a local value *h* of the integration domain. By varying these two parameters one can regulate the obtained approximate solutions in order to get more accurate ones. Therefore, BAO can serve as a sophisticated tool for approximate analysis in various fields of mathematics and mechanics. One of the effective applications of BAO is integrating this operator with the Galerkin method to determine the limit loads for the buckling problem of beams with constant and variable thicknesses. The balance equation of the beam is established from the third-order shear deformation theory. The critical buckling load of the beam is determined by applying BAO to three different cases. The calculation results show that the critical buckling load depends on the boundary conditions and the values of the parameters *h* and *p*. It is shown that BAO can give more accurate approximate solutions than the ones obtained by the convential averaging operator.

Keywords: boundary averaging operator (BAO), Galerkin method, buckling, beam, varying cross--section.

1. INTRODUCTION

Beam structures have long been an interesting topic for many scientists. Beam constructions remain a crucial component in several technological applications such as road and bridge construction, buildings, aircraft, and nuclear facilities. Therefore, the development theories about beams in particular and solid objects in general are focused on research by scientists [1]–[4].

The integral operator is a common and frequently used model in several scientific and technical domains, especially in applied mathematics and mechanics. Integral operators, like Laplace and Fourier transforms, are renowned instances of integral transforms that serve as mappings between two function spaces. Averaging operators are significant within integral operators since they have the ability to amalgamate all values of a function into a single average value. For one-dimensional structures, the typical averaging operator integrates over the overall structural length. For 3-dimensional solids, integration is done over the full volume of the solid. Conventional averaging (CA) is frequently referred to as simple or arithmetic averaging since it assumes that all values play an equal part in the function being considered. Weighted averaging (WA) is an effective strategy that considers each value's unique contribution to the function, in contrast to standard average. Anh Tay Nguyen and colleagues use a dual approach to traditional averaging in order to create a weighted averaging method [5]. An in-depth analysis of a one-parameter weighting function for the unique weighted local averaging (WLA) is introduced. The Galerkin technique with weighted local averaging (GWLA) exemplifies the advantages of using the suggested WLA. The GWLA was used to address the issue of columns buckling under stress, demonstrating that the novel concept may significantly enhance the accuracy of the first-order approximation solution of the Galerkin technique. The analogous linearization approach with weighted averaging was used in [6] to analyze nonlinear vibrating systems.

Recently, a novel method of averaging has been introduced, emphasizing the significance of boundary values [7]. The outcome is a novel boundary averaging operator (BAO) that considers the specific significance of boundary variables. An interesting aspect of BAO is the inclusion of a boundary regulatory parameter, denoted as p, which is influenced by a local value h inside the integration region. By adjusting these two factors, one may control the approximate solutions to achieve more accuracy. The combination of BAO with the Galerkin approach has been shown to be a viable tool for approximating the buckling issue of columns and analyzing the frequency of free vibration in severely nonlinear systems.

This work aims to provide a novel boundary integral operator and demonstrate its use with the Galerkin technique to create an effective approximation tool for analyzing the buckling issue of columns. Efforts have been undertaken to enhance the precision of the first approximation solution of the Galerkin technique in analytical studies, since this solution is often achievable in a straightforward manner. The study by Wang CM, Wang CY, and Reddy [8] provides precise solutions for the buckling of structural elements such as columns, beams, arches, rings, plates, and shells with varying cross-sections, axial forces, and boundary conditions. This work utilizes precise solutions for the buckling issue to validate the correctness of the Galerkin technique using boundary operators.

The paper is structured as outlined below: A novel boundary-averaging operator is introduced in Section 2. BAO is used with the Galerkin technique for the buckling issue of beams in Section 3. Section 4 contains a summary of conclusions and suggestions for further research.

2. BOUNDARY AVERAGING OPERATOR WITH EMPHASIS ON BOUNDARY DOMAINS

Let *g(x)* be an integrable deterministic function of $x \in [0,1]$ and *h* is a local value in *[0,1]*. The conventional average of *g(x)* over the interval *[0,1]* is given by an integral as follows

$$< g(x) >= \int_0^1 \mathrm{g}(x)dx \tag{1}$$

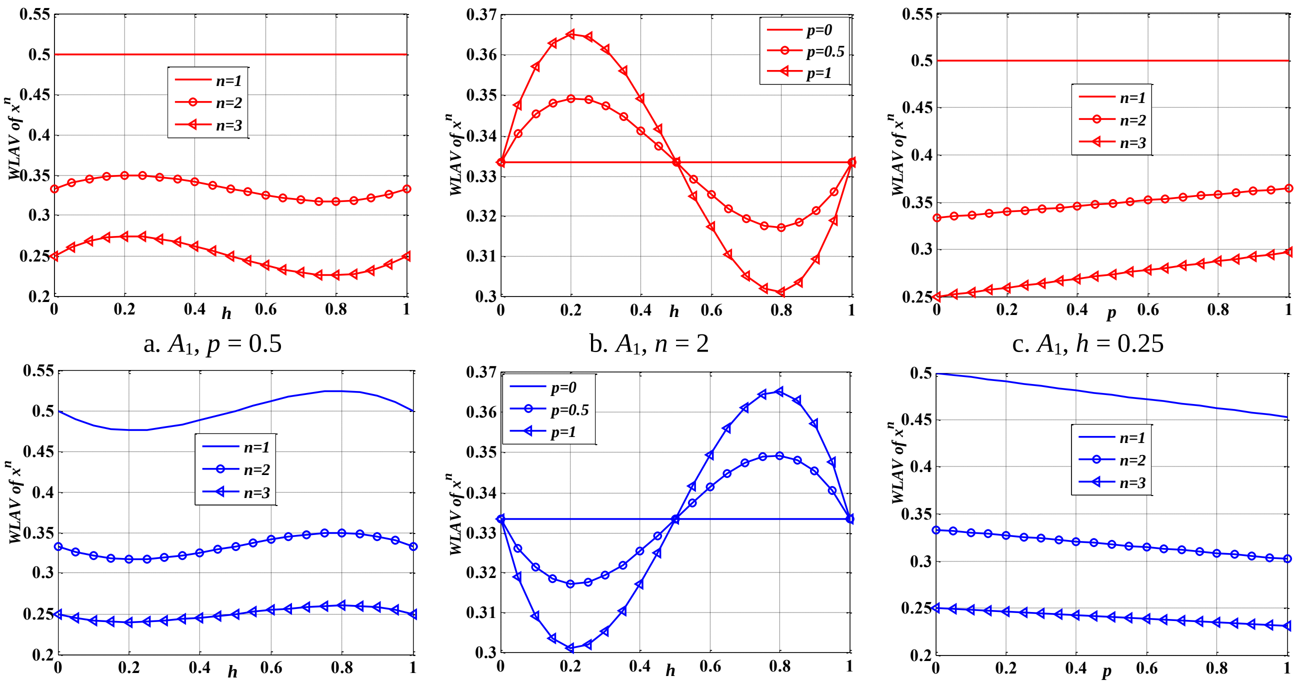
where $<.>$ denotes the conventional averaging operator. The average (1) is called the arithmetic mean because all values of *g(x)* are treated equally and assigned equal weight. However, values of *g(x)* may be weighted for the reason that they belong to different domains of the interval *[0,1]*. It is well known that boundary conditions play a key role in mechanics of solids and structures. To develop this point of view, in addition to the conventional arithmetic average, ND and NT Anh consider the following integral taken over the global domain and over some local boundary domains as follows [7]

$$A_1\,(g(x)) = (1-2ph)\int_0^1 \mathrm{g}(x)dx + p(\int_0^h \mathrm{g}(x)dx + \int_{1-h}^1 \mathrm{g}(x)dx) \tag{2}$$

where the left side is a notation denoting the boundary averaging operator $A_1\,(g(x))$ acting on *g(x)* at a local value *h*, *p* is a weight quoted as a parameter of boundary regulation. The second term in parentheses involves values of *g(x)* integrated into the boundary domains *[0, h]* and *[1-h,1]*. Therefore, the average value (2) can be considered as a weighted average for the reason that the values in the boundary regions are calculated once more and then multiple with a weight *p*. The boundary averaging (2) is coincident with the conventional averaging for *p=0*. In this paper, a new boundary-averaging operator is proposed as follows

$$A_2\,(g(x)) = \int_0^1 \mathrm{g}(x)dx + p[(1-2h)\int_0^h \mathrm{g}(x)dx - h\int_h^{1-h} \mathrm{g}(x)dx] \tag{3}$$

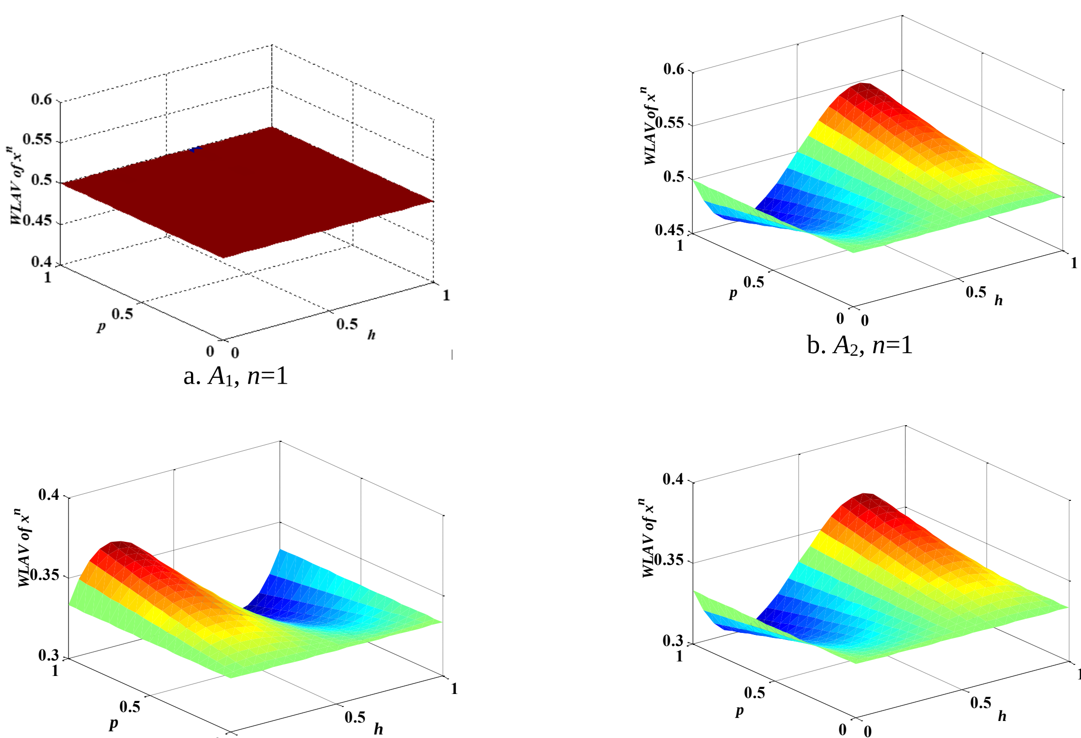
In case the function $\mathrm{g}(x) = x^n$, Figure 1-2 shows the values of A_1-A_2 corresponding to different values of *n*, *p*, and *h*.



a. A_1, p = 0.5 b. A_1, n = 2 c. A_1, h = 0.25

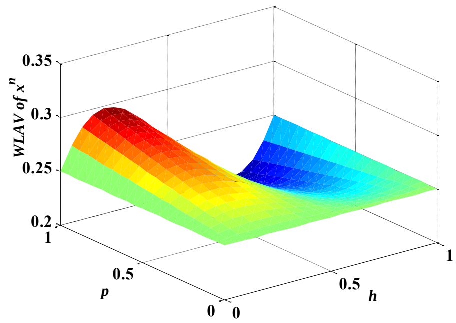
d. A_2, p = 0.5 e. A_2, n = 2 f. A_2, h = 0.25

Fig 1. BAOs (boundary averaging values) of x^n depends on n, p, and h

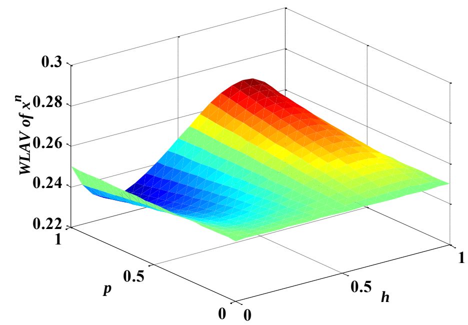


a. A_1, n=1 b. A_2, n=1

d. A_1, n=2 e. A_2, n=2



g. A_1, n=3



h. A_2, n=3

Fig 2. BAOs of x^n depends on h and p for the cases n = 1, n = 2, and n = 3

If function *g(x)* is expanded into Taylor series

$$g(x)=\sum_{i=0}^{\infty} g_i x^i \tag{4}$$

the BAO of *g(x)* can be formulated in its explicit form using the linearity of BAO:

$$A_1: < g(x), h >_p = < \sum_{i=0}^{\infty} g_i x^i, h >_p = \sum_{i=0}^{\infty} g_i < x^i, h >_p = \sum_{i=0}^{\infty} \frac{g_i}{i+1}\left(1+p\left(1-2h+h^{i+1}-(1-h)^{i+1}\right)\right) \tag{5}$$

$$A_2: < g(x), h >_p = < \sum_{i=0}^{\infty} g_i x^i, h >_p = \sum_{i=0}^{\infty} g_i < x^i, h >_p = \sum_{i=0}^{\infty} \frac{g_i}{i+1}\left(1+ph\left(h^i-h^{i+1}-\left(1-h\right)^{i+1}\right)\right) \tag{6}$$

3. APPLICATION OF BOA TO THE BUCKLING PROBLEM OF BEAMS CALCULATED ACCORDING TO HIGH-ORDER SHEAR DEFORMATION THEORY

This work focus beam has a displacement field calculated according to third-order shear deformation theory

$$\begin{cases} u_x(x,y,z) = -z\dfrac{\partial w_b}{\partial x} - f_z\dfrac{\partial w_{zs}}{\partial x} \\ w_z(x,y,z) = w_{zb} + w_{zs} \end{cases} \tag{7}$$

where $f_z = \left(-\dfrac{4}{3}+\dfrac{5z^3}{3h^3}\right)$.

Stress and strain relationship according to Hooke's law

$$\begin{cases} \sigma_x = E\varepsilon_x = E\left(z\varepsilon_z + f_z\varepsilon_f\right);\ \varepsilon_z = -\partial^2 w_{zb}/\partial x^2; \varepsilon_f = -\partial^2 w_{zs}/\partial x^2 \\ \tau = G\gamma_{xz} = Gr_z\gamma_s; \qquad r_z = 1-\partial f_z/\partial z; \gamma_s = \partial w_{zs}/\partial x \end{cases} \tag{8}$$

In the case of axial compression by force $\bar{P}$, the beam's equilibrium equation for the static buckling problem has the form

$$\int_V \delta\varepsilon_x \sigma_x dV = \int_S \bar{P}\left(\partial w_z / \partial w_z\right)\delta\left(\partial w_z / \partial w_z\right) dS \tag{9}$$

After transforming (9), the buckling equation of the beam has a reduced form

$$A_s \frac{\partial^2 \chi}{\partial x^2}\frac{\partial^2 w_{zb}}{\partial x^2} + 2A_s \frac{\partial \chi}{\partial x}\frac{\partial^3 w_{zb}}{\partial x^3} + A_x \frac{\partial^4 w_{zb}}{\partial x^4} = \bar{P}\left\{\left(4\frac{\partial^3 \chi}{\partial x^3} - 2\frac{\partial \chi}{\partial x}\right)\frac{\partial^3 w_{zb}}{\partial x^3} + \left(6\frac{\partial^2 \chi}{\partial x^2} - \chi\right)\frac{\partial^4 w_{zb}}{\partial x^4} - \frac{\partial^2 w_{zb}}{\partial x^2}\right\} \tag{10}$$

where $\chi = -A_x / A_s$, $A_x = \int_{-h_x/2}^{h_x/2} Ez^2 dz$, $A_s = \int_{-h_x/2}^{h_x/2} Gr_z dz$, $G = E/2(1+\nu)$, $r_z = \partial f_z / \partial z.$

By using transformation

$$\bar{x} = x/L,\ w_{zb} = \bar{w}_{zb}/L, P = \bar{P}L^2/EI \tag{11}$$

one gets from Eq. (10)

$$A_s \frac{\partial^2 \chi}{\partial \bar{x}^2}\frac{\partial^2 \bar{w}_{zb}}{\partial \bar{x}^2} + \frac{2A_s}{L}\frac{\partial \chi}{\partial \bar{x}}\frac{\partial^3 \bar{w}_{zb}}{\partial \bar{x}^3} + \frac{A_x}{L^2}\frac{\partial^4 \bar{w}_{zb}}{\partial \bar{x}^4} = \bar{P}\left\{\left(4\frac{\partial^3 \chi}{\partial \bar{x}^3} - 2\frac{\partial \chi}{\partial \bar{x}}\right)\frac{1}{L}\frac{\partial^3 \bar{w}_{zb}}{\partial \bar{x}^3} + \left(6\frac{\partial^2 \chi}{\partial \bar{x}^2} - \chi\right)\frac{1}{L^2}\frac{\partial^4 \bar{w}_{zb}}{\partial \bar{x}^4} - \frac{\partial^2 \bar{w}_{zb}}{\partial \bar{x}^2}\right\} \tag{12}$$

To solve Eq. (12) one needs to add boundary conditions for beam at two points x=0 and x=1. In this paper we consider 3 typical types of boundary conditions as given in Tab. 1. For each type of boundary conditions there exists a corresponding comparison function $\bar{w}_{zb}$ of the polynomial form, $\bar{w}_{zb} = \sum_{i=0}^{n=4} C_i \bar{x}^i$, where C_i are obtained from the boundary conditions. Tab. 1 shows 3 types of boundary conditions and corresponding comparison functions.

Table 1. Different types of beams with corresponding boundary conditions and comparison functions

Type of Boundary Conditions	Boundary Conditions	Comparison Function *W(x)*
1. Pinned-Pinned beam (P-P)	$\bar{w}_{zb}(0) = 0;\ \frac{d^2\bar{w}_{zb}(0)}{d\bar{x}^2} = 0$ $\bar{w}_{zb}(1) = 0;\ \frac{d^2\bar{w}_{zb}(1)}{d\bar{x}^2} = 0$	$\bar{x}^4 - 2\bar{x}^3 + \bar{x}$
2. Clamped-Pinned beam (C-P)	$\bar{w}_{zb}(0) = 0;\ \frac{d\bar{w}_{zb}(0)}{d\bar{x}} = 0$ $\bar{w}_{zb}(1) = 0;\ \frac{d^2\bar{w}_{zb}(1)}{d\bar{x}^2} = 0$	$2\bar{x}^4 - 5\bar{x}^3 + 3\bar{x}^2$
3. Clamped-Clamped beam (C-C)	$\bar{w}_{zb}(0) = 0;\ \frac{d\bar{w}_{zb}(0)}{d\bar{x}} = 0$ $\bar{w}_{zb}(1) = 0;\ \frac{d\bar{w}_{zb}(1)}{d\bar{x}} = 0$	$\bar{x}^4 - 2\bar{x}^3 + \bar{x}^2$

The approximate solution of the buckling problem described by Eq. (12) can be obtained by using Galerkin method for one term comparison function as follows

$$< \left(A_s \frac{\partial^2 \chi}{\partial \bar{x}^2} \frac{\partial^2 \bar{w}_{zb}}{\partial \bar{x}^2} + \frac{2A_s}{L} \frac{\partial \chi}{\partial \bar{x}} \frac{\partial^3 \bar{w}_{zb}}{\partial \bar{x}^3} + \frac{A_x}{L^2} \frac{\partial^4 \bar{w}_{zb}}{\partial \bar{x}^4} - \bar{P} \left\{ \left(4 \frac{\partial^3 \chi}{\partial \bar{x}^3} - 2 \frac{\partial \chi}{\partial \bar{x}} \right) \frac{1}{L} \frac{\partial^3 \bar{w}_{zb}}{\partial \bar{x}^3} + \left(6 \frac{\partial^2 \chi}{\partial \bar{x}^2} - \chi \right) \frac{1}{L^2} \frac{\partial^4 \bar{w}_{zb}}{\partial \bar{x}^4} - \frac{\partial^2 \bar{w}_{zb}}{\partial \bar{x}^2} \right\} \right) \bar{w}_{zb} >= 0 \tag{13}$$

Eq. (13) leads to the approximate buckling load obtained by Galerkin method with CA

$$P_{ca} = - \frac{< \left(A_s \frac{\partial^2 \chi}{\partial \bar{x}^2} \frac{\partial^2 \bar{w}_{zb}}{\partial \bar{x}^2} + \frac{2A_s}{L} \frac{\partial \chi}{\partial \bar{x}} \frac{\partial^3 \bar{w}_{zb}}{\partial \bar{x}^3} + \frac{A_x}{L^2} \frac{\partial^4 \bar{w}_{zb}}{\partial \bar{x}^4} \right) \bar{w}_{zb} >}{< \left\{ \left(4 \frac{\partial^3 \chi}{\partial \bar{x}^3} - 2 \frac{\partial \chi}{\partial \bar{x}} \right) \frac{1}{L} \frac{\partial^3 \bar{w}_{zb}}{\partial \bar{x}^3} + \left(6 \frac{\partial^2 \chi}{\partial \bar{x}^2} - \chi \right) \frac{1}{L^2} \frac{\partial^4 \bar{w}_{zb}}{\partial \bar{x}^4} - \frac{\partial^2 \bar{w}_{zb}}{\partial \bar{x}^2} \right\} \bar{w}_{zb} >} \tag{14}$$

If in Eq. (13) one replaces the conventional averaging by BAO then gets

$$A_1 [\left(A_s \frac{\partial^2 \chi}{\partial \bar{x}^2} \frac{\partial^2 \bar{w}_{zb}}{\partial \bar{x}^2} + \frac{2A_s}{L} \frac{\partial \chi}{\partial \bar{x}} \frac{\partial^3 \bar{w}_{zb}}{\partial \bar{x}^3} + \frac{A_x}{L^2} \frac{\partial^4 \bar{w}_{zb}}{\partial \bar{x}^4} = \bar{P} \left\{ \left(4 \frac{\partial^3 \chi}{\partial \bar{x}^3} - 2 \frac{\partial \chi}{\partial \bar{x}} \right) \frac{1}{L} \frac{\partial^3 \bar{w}_{zb}}{\partial \bar{x}^3} + \left(6 \frac{\partial^2 \chi}{\partial \bar{x}^2} - \chi \right) \frac{1}{L^2} \frac{\partial^4 \bar{w}_{zb}}{\partial \bar{x}^4} - \frac{\partial^2 \bar{w}_{zb}}{\partial \bar{x}^2} \right\} \right) \bar{w}_{zb}] = 0 \tag{15}$$

Then one has the approximate buckling load obtained by Galerkin method with BOA

$$P(p,h) = - \frac{A_1 [\frac{d^2}{dx^2} \left(A_s \frac{\partial^2 \chi}{\partial \bar{x}^2} \frac{\partial^2 \bar{w}_{zb}}{\partial \bar{x}^2} + \frac{2A_s}{L} \frac{\partial \chi}{\partial \bar{x}} \frac{\partial^3 \bar{w}_{zb}}{\partial \bar{x}^3} + \frac{A_x}{L^2} \frac{\partial^4 \bar{w}_{zb}}{\partial \bar{x}^4} \right) \bar{w}_{zb}]}{A_1 [\left\{ \left(4 \frac{\partial^3 \chi}{\partial \bar{x}^3} - 2 \frac{\partial \chi}{\partial \bar{x}} \right) \frac{1}{L} \frac{\partial^3 \bar{w}_{zb}}{\partial \bar{x}^3} + \left(6 \frac{\partial^2 \chi}{\partial \bar{x}^2} - \chi \right) \frac{1}{L^2} \frac{\partial^4 \bar{w}_{zb}}{\partial \bar{x}^4} - \frac{\partial^2 \bar{w}_{zb}}{\partial \bar{x}^2} \right\} \bar{w}_{zb}]} \tag{16}$$

It is seen from Eq. (16) that the critical load determined by Galerkin method with BAO is a function of parameter p and local value h. If the value p is given, the buckling load will be chosen as the lowest value of $P(p,h)$ in the interval *[0.1]* i.e.

$$P_{BAO}(p) = \min_{h \in [0,1]} P(p,h) \tag{17}$$

It is clearly seen from Eqs. (16) and (17) that the approximate buckling load obtained by Galerkin method with CA is corresponding to the case $p=0$ of the approximate buckling load obtained by Galerkin method with BAO.

3.1. Beams with constant cross-section

Tables 2-3 show the buckling loads obtained by Galerkin method with CA ($p=0$) and BAO ($p=0.25$ *and* $p=0.5$) for all 3 types of boundary conditions and compares those with exact values from [8]. The beam has length L, length to-thickness ratio L/h=100, Young's modulus E, and Poisson ratio ν . It should be noted that this work provides a result of calculating the critical buckling load based on the average of three critical buckling load values as follows $P_{BAO}(tb) = \left(P_{BAO}(0) + P_{BAO}(0.25) + P_{BAO}(0.5) \right) / 3$.

Table 2. Accuracy of approximate buckling loads for different types of beam with constant cross-section, calculated according to A_1

Type of *BC*	P_{exact}[8]	$P_{BAO}(0)$	error (%)	$P_{BAO}(0.25)$	error (%)	$P_{BAO}(0.5)$	error (%)	$P_{BAO}(tb)$	error (%)
P-P	9.8696	9.8790	0.10	9.6457	2.27	9.4424	4.33	9.6557	2.17
C-P	20.1907	20.9874	3.95	20.3478	0.78	19.8050	1.91	20.3801	0.94
C-C	39.4784	41.9542	6.27	38.9351	1.38	36.6262	7.22	39.1718	0.78

Table 3. Accuracy of approximate buckling loads for different types of beam with constant cross-section, calculated according to A_2

Type of *BC*	P_{exact} [8]	$P_{BAO}(0)$	error (%)	$P_{BAO}(0.25)$	error (%)	$P_{BAO}(0.5)$	error (%)	$P_{BAO}(tb)$	error (%)
P-P	9.8696	9.8790	0.10	9.7580	1.13	9.6455	2.27	9.7608	1.10
C-P	20.1907	20.9874	3.95	20.4902	1.48	20.0469	0.71	20.5082	1.57
C-C	39.4784	41.9542	6.27	40.3397	2.18	38.9415	1.36	40.4118	2.36

The findings obtained from Tables 2-3 indicate that for beams at the P-P boundary condition, the critical buckling load calculated according to the average value ($P_{BAO}(tb)$) for case A_2 has a smaller error than for case A_1. As for the case of beams with C-P and C-C boundary conditions, the results calculated according to the average value ($P_{BAO}(tb)$) for case A_1 are better than for case A_2.

3.2. Beams with variable cross-section given by the exponential function

In this section, we consider beams with variable cross-section whose moment of inertia is given by exponential function

$$EI(x) = EI_0\, \mathrm{e}^{-kx} \tag{18}$$

The graph of equation (18) is given in Figure 4 for different values of k (k = 0; 0.5; 1.0).

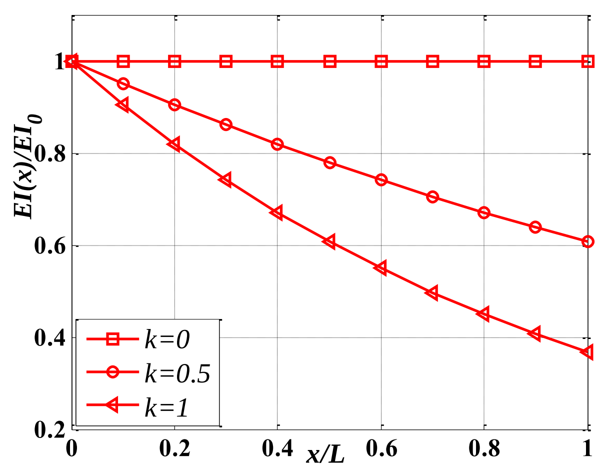


Figure 4. Variation of moment of inertia with different values of exponent *k*

Tabs. 4-9 show the critical buckling loads obtained by BAO for $p = 0.25$ and $p = 0.5$ as well as the ones of CA for 3 types of beams with variable cross-section. The critical buckling load $P_{BAO}(tb)$ for beams with the C-C boundary condition is most accurately calculated using method A_1. When analyzing a beam with the P-P boundary condition, the critical buckling load $P_{BAO}(tb)$ estimated using method A_2 consistently exhibits a lower margin of error compared to the calculation done using method A_1 (unless otherwise k = 1). When analyzing a beam with the C-P boundary condition, the critical buckling load $P_{BAO}(tb)$ obtained using A1 has error that is always smaller than the case using A_2.

Table 4. Accuracy of approximate buckling loads for P-P beam with exponential moment of inertia, calculated according to A_1

k	P_{exact} [8]	$P_{BAO}(0)$	error (%)	$P_{BAO}(0.25)$	error (%)	$P_{BAO}(0.5)$	error (%)	$P_{BAO}(tb)$	error (%)
0	9.8696	9.8790	0.10	9.6457	2.27	9.4424	4.33	9.6557	2.17
0.5	7.6340	7.7089	0.98	7.5366	1.28	7.3864	3.24	7.5440	1.18
1	5.8270	6.0342	3.56	5.9221	1.63	5.8246	0.04	5.9270	1.72

Table 5. Accuracy of approximate buckling loads for C-P beam with exponential moment of inertia, calculated according to A_1

k	P_{exact} [8]	$P_{BAO}(0)$	error (%)	$P_{BAO}(0.25)$	error (%)	$P_{BAO}(0.5)$	error (%)	$P_{BAO}(tb)$	error (%)
0	20.1907	20.9874	3.95	20.3478	0.78	19.8050	1.91	20.3801	0.94
0.5	15.6400	17.5528	12.23	17.0473	9.00	16.6184	6.26	17.0728	9.16
1	11.9900	14.6202	21.94	14.2399	18.76	13.9172	16.07	14.2591	18.92

Table 6. Accuracy of approximate buckling loads for C-C beam with exponential moment of inertia, calculated according to A_1

k	P_{exact} [8]	$P_{BAO}(0)$	error (%)	$P_{BAO}(0.25)$	error (%)	$P_{BAO}(0.5)$	error (%)	$P_{BAO}(tb)$	error (%)
0	39.4784	41.9542	6.27	38.9351	1.38	36.6262	7.22	39.1718	0.78
0.5	30.6000	32.5536	6.38	30.2394	1.18	28.4697	6.96	30.4209	0.59
1	23.4900	25.0608	6.69	23.3460	0.61	22.0351	6.19	23.4806	0.04

Table 7. Accuracy of approximate buckling loads for P-P beam with exponential moment of inertia, calculated according to A_2

k	P_{exact} [8]	$P_{BAO}(0)$	error (%)	$P_{BAO}(0.25)$	error (%)	$P_{BAO}(0.5)$	error (%)	$P_{BAO}(tb)$	error (%)
0	9.8696	9.8790	0.10	9.7580	1.13	9.6455	2.27	9.7608	1.10
0.5	7.6340	7.7089	0.98	7.6118	0.29	7.5215	1.47	7.6141	0.26
1	5.8270	6.0342	3.56	5.9634	2.34	5.8976	1.21	5.9651	2.37

Table 8. Accuracy of approximate buckling loads for C-P beam with exponential moment of inertia, calculated according to A_2

k	P_{exact} [8]	$P_{BAO}(0)$	error (%)	$P_{BAO}(0.25)$	error (%)	$P_{BAO}(0.5)$	error (%)	$P_{BAO}(tb)$	error (%)
0	20.1907	20.9874	3.95	20.4902	1.48	20.0469	0.71	20.5082	1.57
0.5	15.6400	17.5528	12.23	17.1408	9.60	16.7735	7.25	17.1557	9.69
1	11.9900	14.6202	21.94	14.2921	19.20	13.9995	16.76	14.3039	19.30

Table 9. Accuracy of approximate buckling loads for C-C beam with exponential moment of inertia, calculated according to A_2

k	P_{exact} [8]	$P_{BAO}(0)$	error (%)	$P_{BAO}(0.25)$	error (%)	$P_{BAO}(0.5)$	error (%)	$P_{BAO}(tb)$	error (%)
0	39.4784	41.9542	6.27	40.3397	2.18	38.9415	1.36	40.4118	2.36
0.5	30.6000	32.5537	6.38	31.2924	2.26	30.2001	1.31	31.3487	2.45

1	23.4900	25.0609	6.69	24.1030	2.61	23.2736	0.92	24.1458	2.79

4. Conclusions

The averaged values play a key role in many areas of science and engineering hence an extension of these values is presented. For a function given a simple form of weighted local averaging operator (BAO) taking into account the particular role of boundary values of the function is constructed. Remarkable features of BAO are that it contains a parameter of boundary regulation *p* and depends on a local value *h* of the integration domain, and BAO coincides with conventional averaging operator (CAO) at three specific values of *h*, namely $h = 0$; 0.5, and 1. One can regulate the obtained approximate solutions by varying these two parameters to get more accurate ones. In particular, by putting $p = 0$, BAO leads to CAO. It has been shown that the connection of BAO with the Galerkin method forms an effective approximate tool for the buckling problem of beams. Detailed numerical calculations are carried out with three specific values of the boundary regulation parameter, namely, $p = 0.25$; 0.5 for some typical beams. It is obtained that the accuracy of solutions obtained by BAO ($P_{BAO}(tb)$) is significantly improved in comparison to the one of solutions obtained by CA, especially, when the beam has the C-C boundary condition and calculated accordingly A_1. BAO is shown to be an effective tool that is sophisticated and can be supported by CA to obtain more accurate solutions. Further comprehensive investigations, however, need to be carried out in order to find appropriate values of the boundary regulation parameter *p* that can give the most approximate solutions for large classes of problems. The key to the best accuracy of obtained solutions is of course related to the best choice of *p* which will likely depend on the problem to be solved, including the boundary conditions. To solve this problem, the approach of the finite element analysis can be useful.

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