Tymoczko Codes for Standard Young Tableaux

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ABSTRACT. Given a partition λ of integer $n > 0$, there exists a diagram (called Young diagram $\mathscr{Y}_\lambda$) associated with λ. The filling of such diagram from $[n]$ such that the entries increase from top to bottom and from left to right is called the standard Young tableaux (SYT) of shape λ. In this paper, we associate an invariant with each standard Young tableau of shape λ, and provide some combinatorial interpretations of these invariants.

1. INTRODUCTION

Let V be an $n-$dimensional vector space over $\mathbb{C}$, by a flag, we mean a sequence of subspaces $(V_i)_{i=1,\cdots,n}$ ordered by inclusions $V_\bullet = V_1 \subset V_2 \subset \cdots \subset V_n = V = \mathbb{C}^n$, such that $dim_{\mathbb{C}} V_i = i$. The collection of all such flags is called full flag variety denoted by $\mathscr{F}\ell_n(\mathbb{C}) = \{V_\bullet : V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n = \mathbb{C}^n\}$. The set of flags stabilized by a nilpotent operator X (of Jordan type λ) is known as the Springer variety (Spr_λ). T.A. Springer noted that the cohomology ring of this variety carries a symmetric group action in 1976 and provided a thorough geometric formulation of this action. After a decade, Garsia and Procesi in [3], presented the cohomology ring as a graded quotient of a polynomial ring, which improved the clarity and accessibility of Springer's work. In [9], Tymoczko use fillings of Young tableaux to characterize these affine pieces of Springer varieties, and demonstrate that the dimension of the affine piece can be calculated using combinatorial techniques that extend the concept of Eulerian numbers. The study of Springer varieties by Garsia and Procesi was expanded to include a two-parameter generalization of Springer varieties known as Hessenberg varieties in [5].
In her study of the connection between Springer fibers and Schubert varieties, Tymoczko in [8] introduces certain invariants of standard tableaux (SYT) . These are used to construct indexing permutation w_T, called the Schubert point, of Schubert varieties whose union has Betti numbers as a certain Springer fiber in [6].
Using the algorithm through which a permutation was attached to each standard tableau of shape λ, we introduce an invariant, called Tymoczko codes, denoted by (codT). We studied the combinatorial properties of these codes and provided combinatorial interpretations of them. Finally, using the weight associated with the codes, we realize the Bruhat graph of the Schubert variety associated with the set of all standard tableaux of the partitions λ of n.

In section two, we review some basic properties of the symmetric group S_n, partitions, and composition of integers, as relevant to our discussion. In section three, we present and study the combinatorial properties of the Tymoczko code. In section four, we characterize the reduced words associated with each Tymozcko code.

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2. Symmetric Group and Integer Partitions

The symmetric group S_n is generated by the set $S = \{s_1, s_2, \cdots, s_{n-1}\}$ of adjacent transpositions s_i, $(1 \le i \le n-1)$ such that it swaps i and $i+1$ and fixes other elements of $[n]$, subject to braid relations. The length of w, denoted by $\ell(w)$ is the smallest integer $k \ge 0$ such that, w can be written as a product of k elements of S (i.e. $w = s_{c_1} s_{c_2} \cdots s_{c_k} \in S_n$), then, this expression is called the reduced decomposition of w and we say k is the length of w and we write $\ell(w) = k$. The string of subscripts $c_1 c_2 \cdots c_k$ is the word ω of w,(both are not necessarily unique). We shall denote the set of all possible reduced word of $w \in S_n$ by $R(w)$. For details on this topic, readers are encouraged to consult [2],[7] and [4].
A lattice word is a string of integers $a_i > 0$, in which every subword contains at least many a_i as $a_i + 1$.
A Yamanouchi word is a string of positive integers whose reversal is a lattice word.
For instance, string 22233232 is a lattice word, and 23233222 is a Yamanouchi word. Following [1], an increasing factorization for ω, partitions ω into blocks, such that the entries starting from the left increases from left to right within each block.
For instance, the word $\omega = 345231$ is an increasing factorization since it can be factored into blocks 345|23|1 with each block from the left increases from left to right.
For any $w \in S_n$ a reduced factorization for w is an increasing factorization of a reduced word for w.

2.1. The Bruhat order is a partial order $\le$ defined on S_n. For any $\sigma, \tau \in S_n$, we say $\sigma \le \tau$ in Bruhat order if τ can be obtained from σ via a sequence of transpositions. In other words, we say $\sigma \le \tau$ if and only if the reduced word of σ is a subword of the reduced word of τ.

2.2. A partition λ of non negative integer n written as $\lambda \vdash n$, is a sequence $\lambda = (\lambda_i)_{i=1}^k$ of integers such that $\lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_k$ and $\sum_{i=1}^k \lambda_i = n$. Each λ_i is called part of λ. The number of parts is called the length of λ denote by $\ell(\lambda)$, and the sum of parts is the weight of λ denoted by $|\lambda| = \lambda_1 + \lambda_2 + \lambda_3 + \ldots + \lambda_k$. Similar to partition of integers, is a sequence $(a_i)_{i=1}^k$ of nonnegative integers such that $\sum_{i=1}^k a_i = n$ is called composition of nonnegative integer n.
For example, let $n = 4$, the following are all compositions of 4

$$(4),\ (3,1),\ (1,3),\ (2,2),\ (2,1,1),\ (1,2,1),\ (1,1,2),\ (1,1,1,1)$$

We consider $(1,3)$ and $(3,1)$ as different composition but they are the same as partition.

2.3. For any partition λ of an integer $n > 0$, there corresponds a diagram called Young diagram ($\mathscr{Y}_\lambda$) which gives an interesting and pictorial way of visualizing partitions. It is a collection of cells (boxes) arranged in left justified rows such that the number of cells in i^{th} row corresponds to the size of a part λ_i in λ, and is weakly decreasing from top to bottom.
For instance, the Young diagram of $\lambda = (3,2,1)$ is shown in figure below.

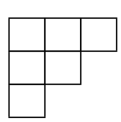


Table 1. Young diagram of shape $\lambda = 3,2,1$

We adopt matrix notation in labeling each cell of $\mathscr{Y}_\lambda$,and we write (i, j) to denote a cell in the i^{th} row and j^{th} columns of $\mathscr{Y}_\lambda$.
We call the filling of $\mathscr{Y}_\lambda$ a row strict tableau (rst) if the filling is such that the entries strictly increase from left to right along the row,with no condition on the columns.

TABLE 2. row strict tableau

1	4	6	8	9	10
3	12	13	14	16	
2	5	15			
7	11				

We denote by $(rst)^\lambda$ the set of row strict tableaux of shape $\lambda = (\lambda_1,\ \lambda_2,\ \cdots,\ \lambda_k)$. The size of $(rst)^\lambda$ denoted by $\#(rst)^\lambda$ is given by the multinomial coefficient. That is,

$$\#(rst)^\lambda = \frac{n!}{\prod_{i=1}^{k} \lambda_i!}$$

If the filling of Young diagram of shape λ is such that the integers from 1 to n appears exactly once and that its entries are increasing across each row and column, then such a filling is call standard Young tableaux.

TABLE 3. standard tableau

1	2	5	6	11	15
3	7	8	12	14	
4	9	16			
10	13				

Remark 2.1. We shall henceforth denote the collection of all standard Young tableaux of shape $\lambda \vdash n$ by $ST_\lambda(n)$, and by $ST(P_\lambda(n))$ the set of all standard Young tableau of all Shapes $\lambda \in P(n)$.

Hook length formula (Frame, Robinson, and Thrall). If λ is a Young diagram with n boxes, then the number $\#ST_\lambda(n)$ of standard tableaux with shape λ is given as

$$\#ST_\lambda(n) = \frac{n!}{\prod_{(i,j)\in\lambda} h_{i,j}}$$

Where $h_{i,j}$ is the number of cells directly to the right and directly bellow the cell in $(i, j)^{th}$ position including the cell.

3. Tymoczko Codes and their Combinatorial Properties

In what follows, we discuss Tymoczko's procedure of associating a permutation to each standard tableaux and then associate a code to this procedure.

Definition 3.1. Let d_i $(1 \leq i \leq n)$, be the number of row(s) above i in $T \in ST_\lambda(n)$, and w_i denote the increasing product of simple transpositions

$$w_i = s_{i-d_i} s_{i-d_i+1} s_{i-d_i+2} \cdots s_{i-2} s_{i-1}$$

where each $s_i = (i, i+1)$. If $d_i = 0$ then $w_i = e$ is the identity. Then the Schubert point associated to T is the permutation $w_T = w_n w_{n-1} w_{n-2} \cdots w_2$ [8]

Example 3.2. Let $n = 6$. Consider a partition $\lambda = (3,2,1)$ with standard tableaux $\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array}$

$d_1 = 0,\ d_2 = 0,\ d_3 = 0,\ d_4 = 1,\ d_5 = 1,\ d_6 = 2$ with $w_6 = s_4 s_5,\ w_5 = s_4,\ w_4 = s_3,\ w_3 = e,\ w_2 = e,$ $w_1 = e,$ then $w_T = s_4 s_5 s_4 s_3$. Arranging the values of the $d_i' s$ in a natural order of $i's$ we obtain an $n-$turple and called it Tymoczko code (denoted by codT) for w_T. For instance, the Tymoczko code in the above example is given as $(d_1,\ d_2,\ d_3,\ d_4,\ d_5,\ d_6) = (0,\ 0,\ 0,\ 1\ ,\ 1,\ 2) =$ cod(T). Let $ST(P_\lambda(n))_{n>2}$ be the set of all the standard tableaux associated with shapes λ where λ are the partitions of n.
Define a map

$$\phi : ST(P_\lambda(n)) \longrightarrow \mathbb{Z}^n$$

by

$$T \mapsto \text{cod(T)}.$$

Which takes the standard tableaux T to the $n-$tuple $(d_1, \cdots, d_n)$ of integer vectors, where d_i is the number rows strictly above i in T, and denote it by cod(T). We call $(d_1, \cdots, d_n)$ the Tymoczko code associated to T.

Example 3.3. Let $n = 6$ T ∈ ST(P(6)) such that T is of shape $\lambda = (3,2,1)$. There are five of such $T \in ST(P(6))$. Consider $T = \begin{array}{|c|c|c|}\hline 1 & 4 & 6 \\ \hline 2 & 5 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array}$ then codT $= (0,1,2,0,1,0)$.

We attach a word to codeT denoted by $\omega(T)$ by eliminating the brackets and comas between the coordinates of codT. In the above example, we have $\omega(T) = 012010$

Theorem 3.4. *Let A_r be the collection of the integer coordinates d_i in* codT *such that $d_i = r$ and let $\lambda = (\lambda_1, \cdots, \lambda_{\ell(\lambda)})$ be the shape of* T. *Then*

i) *A_r is either a singleton set or multiset.*
ii) *The size of A_r is λ_{1+r}, $r \in [0,\ \ell(\lambda) - 1]$.*

Proof.

i) Since A_r is the collection of all d_i such that $d_i = r$ and d_i is the number of rows strictly above i in T, then the size of A_r will definitely be one, if it happens that i is the only entry in the $(r+1)^{th}$ row, otherwise it is a multiset.

ii) We show the second part of the theorem is true by induction on r starting from $r = 0$. For $r = 0$, we have

$$A_0 = \{d_i \mid d_i = 0\}$$

The implication of this is that, all $i's$ such that $d_i = 0$ appear in the cells of the first row of T from the top, and the number of cells in this first row is determined by λ_1. Hence $\#A_0 = \lambda_{1+0} = \lambda_1$.
For $r = 1$.
Here,

$$A_1 = \{d_i \mid d_i = 1\}$$

As it is in the case of $r = 0$, the number of rows strictly above i is 1. This implies that all $i's$ such that $d_i = 1$ are in the second row of T (since T is a standard tableaux) and the number of entries in the second row of T is determined by λ_2
Therefore, $\#A_1 = \lambda_2 = \lambda_{1+1}$.
Now, for an arbitrary value of $r = k > 0$, we have

$$A_k = \{d_i : d_i = k\}$$

The number of rows strictly above i is k and all such $i's$ are in the $(k+1)^{th}$ row of T, where the number of entries in that row is determined by λ_{1+k}.
Hence $\#A_k = \lambda_{1+k} = \lambda_{1+k}$.
Finally, we consider the case of $r = \ell(\lambda) - 1$ where we have

$$A_{\ell(\lambda)-1} = \{d_i \mid d_i = \ell(\lambda) - 1\}$$

Entries i that satisfy the condition in $A_{\ell(\lambda)-1}$ appear in the last row (bottom) of T. We know that the number of such entries are determined by $\lambda_{\ell(\lambda)}$ and the number of rows strictly above those entries is determined by $\ell(\lambda) - 1$. Then we have
$\#A_{\ell(\lambda)-1} = \lambda_{\ell(\lambda)} = \lambda_{1+\ell(\lambda)-1}$

□

Example 3.5. Let $n = 7$ and $\lambda = (3,2,2)$, $\ell(\lambda) = 3$. Consider

$$T = \begin{array}{|c|c|c|}\hline 1 & 4 & 7 \\ \hline 2 & 5 \\ \cline{1-2} 3 & 6 \\ \cline{1-2}\end{array}$$

with $d_1 = 0,\ d_2 = 1,\ d_3 = 2,\ d_4 = 0,\ d_5 = 1,\ d_6 = 2,\ d_7 = 0$, then,
$A_0 = \{d_1,\ d_4,\ d_7\}$, $\#A_0 = 3 = \lambda_1$, $A_1 = \{d_2,\ d_5\}$, $\#A_1 = 2 = \lambda_2$, and $A_2 = \{d_3,\ d_6\}$, $\#A_2 = 2 = \lambda_3$

Remark 3.6. So, given a partition $\lambda = (\lambda_1, \cdots, \lambda_{\ell(\lambda)})$ of n, the Tymoczko code (codT) associated with the standard tableau T of shape λ, has λ_i integer coordinates $i - 1$ where $1 \le i \le k$. It turns out that the values of the coordinate of codT encodes the partition λ.

Lemma 3.7. *Tymoczko code associated to a standard tableau is a lattice word.*

Proof. We recall that a Young diagram ($\mathscr{Y}_\lambda$) of shape λ is a pictorial view of partition $\lambda = \lambda_1 \geq \cdots \lambda_k$, and we are considering $\mathrm{T} \in \mathrm{ST}(\mathrm{P}_\lambda(\mathrm{n}))$.
Therefore, the number of $d_i = 0,\ 1,\ \cdots,\ k-1$, are respectively determine by $\lambda_1, \cdots, \lambda_k$.
Since $\lambda_1 \geq \lambda_j$, $2 \leq j \leq k$ and $d_i = 0$, implies that there are $\lambda_1\ 0's$ in $\omega(\mathrm{T})$, hence there will be at least many $0's$ in any subword of $\omega(\mathrm{T})$ as $1's$.
Also, since $\lambda_2 \geq \lambda_l$, $3 \leq l \leq k$ and $d_i = 1$, implies that there are $\lambda_2\ 1's$ in $\omega(\mathrm{T})$, hence there will be at least many $1's$ in any subword of $\omega(\mathrm{T})$ as $2's$.
In general, for any $v \geq 1$, $\lambda_v \geq \cdots \geq \lambda_k$, $d_i = v-1$ implies that the number of $(v-1)$ in $\omega(\mathrm{T})$ is λ_v, and this leads to at least many occurrence of $(v-1)'s$ as $v's$.
Hence $\omega(\mathrm{T})$ is a lattice word. $\square$

In what follows, we use the weight associated to each code $\mathrm{ST}(\mathrm{P}_\lambda(\mathrm{n}))$ to realize the Bruhat graph of the Schubert points associated with the set of all standard Young tableaux of all shapes $\lambda \in P(n)$.

It will be noticed from the graph that for $n \geq 2$, the number of $\mathrm{T} \in \mathrm{ST}(\mathrm{P}_\lambda(\mathrm{n}))$ with minimal weight (0) is one, the number of $\mathrm{T} \in \mathrm{ST}(\mathrm{P}_\lambda(\mathrm{n}))$ with maximal weight is also one. Of important interest to us at this point, is the number of $\mathrm{T} \in \mathrm{ST}(\mathrm{P}_\lambda(\mathrm{n}))$ with weight one and there are $n-1$ of such $\mathrm{T} \in \mathrm{ST}(\mathrm{P}_\lambda(\mathrm{n}))$. These happens to be the generators of codT of other weights. For instance,consider $n = 5$, below is a table of all $\mathrm{T} \in \mathrm{ST}(\mathrm{P}_\lambda(5))$ and the corresponding weight.

$wt(T)$	$\mathrm{T} \in \mathrm{ST}(\mathrm{P}_\lambda(5))$
10	$\ytableaushort{1,2,3,4,5}$
6	$\ytableaushort{12,3,4,5}$ $\ytableaushort{13,2,4,5}$ $\ytableaushort{14,2,3,5}$ $\ytableaushort{15,2,3,4}$
4	$\ytableaushort{12,34,5}$ $\ytableaushort{13,24,5}$ $\ytableaushort{14,25,3}$ $\ytableaushort{12,35,4}$ $\ytableaushort{13,25,4}$
3	$\ytableaushort{123,4,5}$ $\ytableaushort{124,3,5}$ $\ytableaushort{125,3,4}$ $\ytableaushort{134,2,5}$ $\ytableaushort{135,2,4}$ $\ytableaushort{145,2,3}$
2	$\ytableaushort{123,45}$ $\ytableaushort{124,35}$ $\ytableaushort{134,25}$ $\ytableaushort{135,24}$; $\ytableaushort{125,34}$
1	$\ytableaushort{1234,5}$ $\ytableaushort{1235,4}$ $\ytableaushort{1245,3}$ $\ytableaushort{1345,2}$
0	$\ytableaushort{12345}$

TABLE 4. Standard Tableaux of all Partition of 5 and their Corresponding Weight

Proposition 3.8. *For any* $\mathrm{T} \in \mathrm{ST}(\mathrm{P}_\lambda(\mathrm{n}))$, codT *can be uniquely expressed as a linear combination of codes with weight one.*

Proof. Given any $\mathrm{T} \in \mathrm{ST}(\mathrm{P}_\lambda(\mathrm{n}))$, there are n coordinates in codT with the first coordinate always equal to zero. Therefore, there are $n-1$ coordinates which are either zero or a, $1 \le a \le \ell(\lambda)-1$.
For T with weight one, there are $n-1$ of them with $\mathrm{codT} = \mathrm{e_i}$, $2 \le \mathrm{i} \le \mathrm{n}$, where e_i is an $n-$tuple with 1 in i^{th} position and zero elsewhere. Hence, the result follows from the elementary linear algebra that every $\mathbf{x} \in \mathbb{R}^n$ can be uniquely expressed as a linear combination of e_i. ☐

Example 3.9. Let $n = 5$ and $\lambda = (2,2,1)$. Consider $\mathrm{T} \in \mathrm{ST}(\mathrm{P}_\lambda(5))$, such that $\mathrm{T} = \ytableaushort{12,34,5}$. Then

$$\mathrm{codT} = (0,0,1,1,2) = 0(0,0,0,0) + 0(0,1,0,0,0) + 1(0,0,1,0,0) + 1(0,0,0,1,0) + 2(0,0,0,0,1).$$

The polynomial corresponding to the weights of cod(T) for all $\mathrm{T} \in \mathrm{ST}(\mathrm{P}_\lambda(5))$ in table 3 is

$$\mathrm{P(wt(T),x)} = 1 + 4\mathrm{x} + 5\mathrm{x}^2 + 6\mathrm{x}^3 + 5\mathrm{x}^4 + 4\mathrm{x}^6 + \mathrm{x}^{10}$$

The coefficients of each term is the number of $\mathrm{T} \in \mathrm{ST}(\mathrm{P}_\lambda(\mathrm{n}))$ whose weights give the index of x in the term. For $1 \le n \le 6$, the associated polynomial $\mathrm{P}(\mathrm{wt}(\mathrm{T}),\mathrm{x})$ is palindromic, this we display in the table below.

n	$\mathrm{P}(\mathrm{wt}(\mathrm{T}),\mathrm{x})$
1	1
2	$1+\mathrm{x}$
3	$1+2\mathrm{x}+\mathrm{x}^3$
4	$1+3\mathrm{x}+2\mathrm{x}^2+3\mathrm{x}^3+\mathrm{x}^6$
5	$1+4\mathrm{x}+5\mathrm{x}^2+6\mathrm{x}^3+5\mathrm{x}^4+4\mathrm{x}^6+\mathrm{x}^{10}$
6	$1+5\mathrm{x}+9\mathrm{x}^2+15\mathrm{x}^3+16\mathrm{x}^4+15\mathrm{x}^6+9\mathrm{x}^7+5\mathrm{x}^{10}+\mathrm{x}^{15}$
7	$1+6\mathrm{x}+14\mathrm{x}^2+29\mathrm{x}^3+35\mathrm{x}^4+21\mathrm{x}^5+41\mathrm{x}^6+35\mathrm{x}^7+14\mathrm{x}^9+15\mathrm{x}^{10}+14\mathrm{x}^{11}+6\mathrm{x}^{15}+\mathrm{x}^{21}$

TABLE 5. Table of polynomials corresponding to the weights all $\mathrm{T} \in \mathrm{ST}(\mathrm{P}_\lambda(\mathrm{n}))$

4. CHARACTERIZATION OF SCHUBERT POINTS w_T ASSOCIATED TO STANDARD YOUNG TABLEAUX

Here, we study the composition structure of the reduced word of Schubert points w_T and describe its standard form for any λ. this is displayed in our next result present the canonical form for the structure of the reduced word of w_T

Proposition 4.1. *Let w_T be the Schubert point associated to $\mathrm{T} \in \mathrm{ST}_\lambda(\mathrm{n})$ of any shape, with $\mathrm{codT} = (\mathrm{d}_1, \mathrm{d}_2, \cdots, \mathrm{d}_\mathrm{n})$. Then, the standard form for the composition structure of the reduced word of w_T is given as*
$a_1(a_1+1)(a_1+2)\cdots(a_1+k_1)|a_2(a_2+1)(a_2+2)\cdots(a_2+k_2)|\cdots|a_r(a_r+1)(a_r+2)$
$\cdots(a_r+k_r)|$
Where $a_j = (i-d_i)$, $k_j = d_i - 1$ and $j = n-i+1$, $1 \le j \le r$, r is the number of d_i such that $d_i \ne 0$, $1 \le i \le n$.

Proof. Let $w_T \in \mathrm{ST}_\lambda(\mathrm{n})$ such that T is of any shape λ.
Let $j = n-i+1$. Suppose $d_i = 0$, then there is nothing to proof since w_i, $(2 \le i \le n)$ is always an identity (from the definition of w_i). Now, suppose $d_i \ne 0$ and $i = n$. Then $j = n-n+1$ which implies that $a_1 = (n-d_n)$.
Since $d_n \ne 0$, let's assume $d_n = q$, $1 \le q \le \ell(\lambda)-1$.
From the definition of w_i in [8], $w_n = s_{n-q}s_{n-q+1}s_{n-q+2}\cdots s_{n-2}s_{n-1}$, then the first block from the left is written as

$$|(n-q)(n-q+1)(n-q+2)\cdots(n-2)(n-1)|$$

By replacing n with i and q with d_i in the above, we have

$$|(i-d_i)(i-d_i+1)(i-d_i+2)\cdots(i-d_i+d_i-2)(i-d_i+d_i-1)|$$

with $a_j = (i-d_i)$ and $k_j = d_i - 1$ then the above equation becomes

$$|a_j(a_j+1)(a_j+2)\cdots(a_j+k_j-1)(a_j+k_j)|$$

Also, we have from the theorem that $j = n-i+1$ which implies that $j = 1$ (since $i = n$ by hypothesis). Hence, we have $|a_1(a_1+1)(a_2+2)\cdots(a_1+k_1-1)(a+k_1)|$ as the first block of the

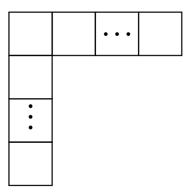
composition structure of w_T provided $d_n \neq 0$.
By mimicking the proof of the first block we obtain the structure of the remaining blocks. □

Example 4.2. Let codT $= (0,0,0,1,1,2)$, be a code of a certain Schubert point . It is easy to see that the shape of the associated partition is $\lambda = (3,2,1)$ with $n = 6$. From the statement of the theorem, we have that; $j = n - i + 1,\ 1 \leq j \leq 3,\ a_j = (i - d_i),\ k_j = d_i - 1$. Now, when $i = n = 6$, then $j = 1 \implies,\ a_1 = 4$ also, $k_1 = 1$. Therefore we have $a_1(a_1 + k_1)| = 45|$. This give the first block. For the second and third block, we respectively have $i = 5$ and $i = 4,\ a_2 = 4,\ k_2 = 0$ which implies that 4 is the only element in the second block. and for the third block $a_3 = 3$. Hence $a_1(a_1+1)|a_2|a_3 = 45|4|3$ is the composition structure of the given code. We confirm this by computing the Schubert point $w_T = s_4s_5s_4s_3$ of the standard tableau $T = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2} 6 \\ \cline{1-1}\end{array}$ associated to the given code.

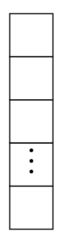
Theorem 4.3. *Let $\ell(\alpha_{c_T})$ be the length of the partition associated to the composition structure of Schubert point w_T identified with standard tableau of shape λ. Then*

(i) $\ell(\lambda) = \ell(\alpha_{c_T}) + 1$ *if* $\lambda = (n-k,\ 1^k),\ k \geq 2$.
(ii) $\ell(\lambda) = \ell(\alpha_{c_T})$ *if* $\lambda = (n-(k+2), 2, 1^k),\ n \geq 4,\ 0 \leq k \leq n-4$.

Proof. (i) In this case, the corresponding Young diagram is either of the form



or



depending on either $2 \leq k < n$ or $k = n$ respectively. For each T $\in \mathrm{ST}_\lambda$(n), we define $\mathrm{D_T} = \{\mathrm{d_i} \mid \mathrm{d_i} \neq 0\}$.
Obviously, $\#D_T =$
$ell(\lambda) - 1$, since $d_i = 0$ for all i in the first row of T (from the top). Recall that each $d_i \neq 0$ determines a block in the reduced word of w_T.
This implies that

$$\ell(\alpha_{c_T}) = \#D_T$$

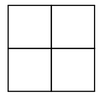
and

$$\ell(\alpha_{c_T}) = \ell(\lambda) - 1$$

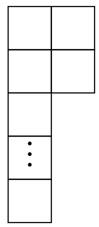
Hence

$$\ell(\alpha_{c_T}) + 1 = \ell(\lambda)$$

(ii) Here, the corresponding Young diagram is either of the form



, if $n = 4$ and $k = 0$ or



if $n > 5$ and k increases with the value of n. Now, suppose the length of the first column is r, then $\alpha_{c_T} = (r-1, r-2, \cdots, 1, 1)$. Which implies $\ell(\alpha_{c_T}) = (r-1)+1 = r$. We recall that the length of the first column of a Young diagram of shape λ is equal to $\ell(\lambda)$. In other words $\ell(\lambda) = r$. Hence $\ell(\lambda) = \ell(\alpha_{c_T})$.

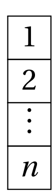
□

Theorem 4.4. *Let* $\alpha_{c_T} = (c_1,\ c_2,\ \cdots,\ c_k)$ *be the composition structure associated with* w_T*, then* $c_k = 1$ *for all* $\mathrm{T} \in \mathrm{ST}_\lambda(\mathrm{n})$.

Proof. It is known from [8] that each $w_T \in S_n$ is of the form $w_T = w_n w_{n-1} w_{n-2} \cdots w_2$ where each $w_i = s_{i-d_i} s_{i-d_{i+1}} s_{i-d_{i+2}} \cdots s_{i-2} s_{i-1},\ 2 \le i \le n$. The reduced word of each w_i gives a block (since the subscripts are increasing in a natural order),and hence forms a part in α_{c_T}.
Assume $i = 2$, then $w_2 = s_{2-d_2}$. In this case $d_2 = 1$, this implies $w_2 = s_1$ hence $c_k = 1$
If $d_2 = 0$, then 2 must be in the first row of T and w_T becomes $w_T = w_n w_{n-1} w_{n-2} \cdots w_3$ with $d_3 = 1$ or 2. If $d_3 = 2$ then 1 and 2 must be above 3 in the same column which implies that $d_2 = 1$, otherwise $d_3 = 1$ n n $w_3 = s_2$ which implies that $c_k = 1$.
Finally, assume $w_T = w_n w_{n-1} w_{n-2} \cdots w_r,\ \ r > 3$, then $w_r = s_{r-d_r} s_{r-d_{r+1}} s_{r-d_{r+2}} \cdots s_{r-2} s_{r-1}$ which implies that there are some $q < r$ above r in the same column which in not possible.
Therefore, $w_r = s_{r-d_r}$ with $d_r = 1$. Hence $c_k = 1$ □

Lemma 4.5. *Let* λ *be a partition of* $n > 0$ *such that* $\lambda = 1^n$. *Consider the Schubert point* w_T *of standard Young tableau of shape* λ *with* $\alpha_{c_T} = (c_1, c_2 \cdots c_k)$ *being the partition associated to the composition structure of* $w_T \in S_n$. *Then* α_{c_T} *is always a staircase partition with* $c_i = n - i$ *and* $\sum_{i=1}^{n-1} c_i = \binom{n}{2}$.

Proof. Let $\lambda = 1^n,\ n \ge 2$, there is only one standard tableau in this case, and is of the form



with w_T of the form $w_T = s_1 s_2 \cdots s_{n-1} s_1 s_2 \cdots s_{n-2} \cdots s_1 s_2 s_1$ with the word (string of subscript) $(123 \cdots n-1)(123 \cdots n-2) \cdots (12)(1)$, arranging the cardinalities of the blocks in descending order, we have $\alpha_{c_T} = (n-1, n-2, \cdots, 2, 1)$

To show that $\sum_{i=1}^{n-1} c_i = \binom{n}{2,}$ We note that

$$\begin{aligned}\sum_{i=1}^{n-1} c_i &= (n-1) + (n-2) + \cdots + (n-(n-1)) \\ &= (n-1) + (n-2) + \cdots + 1 \\ &= n(n-1) - (1+2+3+\cdots+(n-1))\end{aligned}$$

Recall that the sum of the first $n-1$ natural numbers is given as $\frac{n(n-1)}{2}$. Therefore, equation 4.22 becomes

$$= n(n-1) - \frac{n(n-1)}{2} = \frac{n(n-1)}{2} = \binom{n}{2}$$

□

Since our dear Schubert points w_T, $T \in ST(P_\lambda(n))$ are always elements of S_n, expressed in its reduced decompositions (which are not unique). A big question begging for answer here is that, which of the reduced decompositions of $w \in S_n$ gives w_T? We give answer to this question in our next remark

Remark 4.6. Let $w_T \in S_n$, for any $v \in R(w_T)$ to be a Schubert point,It must satisfies the following conditions:

i) v must be a reduced factorization,
ii) v should be able to generate a code such that its first coordinate is zero.
iii) v should be able to generate a tableaux such that the number of $d_i = r, (0 \leq r \leq \ell(\lambda) - 1, 1 \leq i \leq n)$ must be equal to λ_{r+1}

Below is the sage command for :

(1) generating the code of any standard Young tableau with its weight and command for generating the associated Standard Young tableau given any code.
(2) computing the polynomial associated to the set of standard tableaux of all partitions of $n > 0$,
(3) determining the degree of the polynomials in 2.

5. Conclusion

Springer varieties (Spr_λ), also known as Springer fibers, Where λ is a partition of integer $n > 0$, are subvarieties of the full flag varieties FV. The geometry and combinatorics of Springer varieties has been an active area of research over decades.

In the study of the connection between Springer varieties and another subvarieties of $\mathscr{F}\ell_n\mathbb{C}$ called Schubert varieties, Tymoczko in [8] introduces certain algorithm through which she attached a permutation w_T, called the Schubert point, to each row-strict tableaux of shape λ, whose union has Betti numbers as a certain Springer varieties in [6]. The length $\ell(w_T)$ of these permutations turns out to be equal to the dimension of T which was equally introduced by Tymoczko in [9]

Through the algorithm introduce in [8], invariants were attached to a set of row- strict tableaux of shape λ, the attached invariants were studied, investigated and give some combinatorial properties of these invariants and re-interpret some of the results in [8] and [6] in terms of these properties. Lastly, using the weight associated with the codes, Bruhat graph of the Schubert variety associated with the set of all standard tableaux of the partitions λ of n was realised.

Sage Code Implementation

FIGURE 1. Sage Implementation.

January 12, 2023

```
In [1]: sage: def code_to_Tableau(C): # This function the Standard tableaux of a given code.
        ....:     L =[]
        ....:     T = []
        ....:     for i in C:
        ....:         if (i in L)==False:
        ....:             L.append(i)
        ....:     for j in L:
        ....:         H = []
        ....:         t =0
        ....:         for k in C:
        ....:             t = t+1
        ....:             if j==k:
        ....:                 H.append(t)
        ....:         T.append(H)
        ....:     return Tableau(T)
        ....:
        ....: def code_weight(C): # This function returns the weight of a given code
        ....:     return sum(C)
        ....:
        ....: def tableau_to_Code(T): # This function return the code of a given code
        ....:     C = range(sum(T.shape()))
        ....:     for i in range(len(T)):
        ....:         for j in T[i]:
        ....:             C[j-1] = i
        ....:     return (C)

In [2]: sage: T = Tableau([[1,3,5],[2,6],[4]])
        sage: T1 = Tableau([[1,3,5],[2,6],[4]])
        sage: C1 = tableau_to_Code(T1)
        sage: C1

Out[2]: [0, 1, 0, 2, 0, 1]

In [3]: sage: code_weight(C1)

Out[3]: 4

In [4]: sage: code_to_Tableau(C1)
```

Out[4]: [[1, 3, 5], [2, 6], [4]]

In [8]:
```
sage: T= Tableau([[1,2,3,4],[5,6,7],[8]])
sage: T2= Tableau([[1,2,3,4],[5,6,7],[8]])
sage: C2= tableau_to_Code(T2)
sage: C2
```

Out[8]: [0, 0, 0, 0, 1, 1, 1, 2]

In [10]:
```
sage: code_weight(C2)
```

Out[10]: 5

In [11]:
```
sage: code_to_Tableau(C2)
```

Out[11]: [[1, 2, 3, 4], [5, 6, 7], [8]]

In [12]:
```
def tableau_weight(T):
    weight =0
    for i in range(len(T)):
        weight += i*T[i]
    return weight

def     weight_poly(n):
        P =Partitions(n)
        R.<x> = PolynomialRing(QQ, order='lex')
        poly = 0
        for i in list(P):
            coef =StandardTableaux(i).cardinality()
            exp = tableau_weight(i)
            poly += coef*x^exp
        return poly
```

In [13]:
```
weight_poly(1)
```

Out[13]: 1

In [14]:
```
weight_poly(2)
```

Out[14]: x + 1

In [15]:
```
weight_poly(3)
```

Out[15]: x^3 + 2*x + 1

In [16]:
```
weight_poly(4)
```

Out[16]: x^6 + 3*x^3 + 2*x^2 + 3*x + 1

In [17]:
```
weight_poly(5)
```

```
Out[17]: x^10 + 4*x^6 + 5*x^4 + 6*x^3 + 5*x^2 + 4*x + 1

In [18]: weight_poly(6)

Out[18]: x^15 + 5*x^10 + 9*x^7 + 15*x^6 + 16*x^4 + 15*x^3 + 9*x^2 + 5*x + 1

In [5]: def     poly_deg(n):
                deg=binomial(n,2)
                return deg

In [6]: poly_deg(1)

Out[6]: 0

In [7]: poly_deg(2)

Out[7]: 1

In [8]: poly_deg(3)

Out[8]: 3

In [9]: poly_deg(4)

Out[9]: 6

In [10]: poly_deg(5)

Out[10]: 10

In [11]: poly_deg(6)

Out[11]: 15

In [13]: poly_deg(7)

Out[13]: 21

In [14]: poly_deg(8)

Out[14]: 28

In [ ]:
```

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