Approximation of Functions in L_(p,C_n) (X)-space

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Abstract

This research focused on finding a mathematical space, extending to the Lebesgue spaces, through which it is possible to find the best approximation for unbounded functions depending on the fundamental theory of approximation(Korevkin Theorem) using some linear operators in terms of the averaged modulus of smoothness of order k (τ-modulus).

Key words: Multiplier convergence, Multiplier integral, Multiplier modulus.

1. Introduction

In 1949, G. Hardy **[1]** defined the multiplier sequence for a converge of the series as.

A series $\sum_{n=0}^{\infty} c_n$ is called a multiplier convergent if there is convergent sequence of real numbers $\{\Phi_n\}_{n=0}^{\infty}$,such that $(\sum_{n=0}^{\infty} c_n \;\; \Phi_n < \infty)$ where, $\{\Phi_n\}_{n=0}^{\infty}$ is called a multiplier for the convergence, for example***.***

The series $\sum_{m=1}^{\infty} \frac{1}{m}$ is a divergent series and the sequence $\left\{\frac{1}{m}\right\}_{m=1}^{\infty}$ convergent sequence. Since $\sum_{m=1}^{\infty} \frac{1}{m} \cdot \frac{1}{m} = \sum_{m=1}^{\infty} \frac{1}{m^2}$ which is convergent series then the series $\sum_{m=1}^{\infty} \frac{1}{m}$ is a multiplier convergent, so from above we note that.

If $\sum a_n$ is convergent series then it is multiplier convergent,

this by Taken $\{ \emptyset_n\}_{n=0}^{\infty} = \{1\}_{n=0}^{\infty}$.

Similar to the above we provide the following definition.

Let $L_p(\mathbb{X})$ be the space of all bounded measurable functions defined on $\mathbb{X} = [a, b]$ with the norm

$$\|f(.)\|_{Lp} = \|f(.)\|_p = (\int_{\mathbb{X}} |f(x)|^p)^{\frac{1}{p}} < \infty \quad ,1 \le p < \infty.$$

Now for any real valued function f defined on $B = [a, b]$, if there is a sequence $\{ \mathcal{C}_n\}_{n=0}^{\infty}$ of real numbers such that $\int_B (f\mathcal{C}_n)(x) < \infty$, as $n \to \infty$, then f is called multiplier integral and $\{\mathcal{C}_n\}_{n=0}^{\infty}$, is called a multiplier for the integral where $n \in \mathbb{N}$.

The multiplier integral norm can be defined as follows

$$\|f(.)\|_{L_{p,\mathcal{C}_n}} = \left\{(\int_{\mathbb{X}} |(f\mathcal{C}_n)(x)|^p \, dx)^{\frac{1}{p}} : x \in \mathbb{X}\right\} ,$$

Where $\mathcal{C}_n = \{\mathcal{C}_n\}_{n=0}^{\infty}$ be the multiplier for the integral and $\|f\|_{L_{p,\mathcal{C}_n}} = \|f\|_{p,\mathcal{C}_n}$.

From above.

Let $L_{p,\mathcal{C}_n}(\mathbb{X}) = \{f: \|f\|_{p,\mathcal{C}_n} < \infty\}$, $\mathbb{X} = [0,\infty)$, be the space of all unbounded continuous functions$f$, $1 \le p < \infty$, which are equipped with the above norm.

Where $(f\mathcal{C}_n)$ is the sequence of real continuous functions on $[0,\infty)$ and

$\|f\mathcal{C}_n(.)\|_p = \|f(.)\|_{p,\mathcal{C}_n}$ (1.1)

Now After this introduction, we present the definition of the Szasz-Mirkjan-Beta operators . The classical Szasz-Mirkjan operators are defined by

$\varsigma_n(f,x) = \sum_{k=0}^{\infty} j_{n,k}(x) f\left(\frac{k}{n}\right)$ where $j_{n,k}(x) = e^{-nx}\frac{(nx)^k}{k!}, n \ and \ k \in \mathbb{N}, x \in [0,\infty)$.

Some authors like in **[2]** had been proposed a sequence of mixed summation integral type operators, the so called Szasz-Mirakjan-Beta operators as follows:

$$\mathrm{M}_n(f,x) = e^{-nx} f(0) + e^{-nx} \sum_{k=1}^{\infty} \frac{(nx)^k}{k!\,\beta(n+1,k)} \int_0^{\infty} f(t) \frac{t^{k-1}}{(1+t)^{n+k+1}} dt$$

Where $f \in C[0,\infty), x,t \in [0,\infty)$ and

$$\beta(n+1,k) = \int_0^{\infty} \frac{t^{k-1}}{(1+t)^{n+k+1}} dt \quad (\text{Beta} - \text{Function})$$

Another authors like in **[4]** Provide a Modified Szasz-Mirkjan-Beta Operators as follows:

For $f \in \boldsymbol{C_y}[0,\infty)$ and $n \in \mathbb{N}$ then

$$M_n(f,x) = e^{-nx} \sum_{k=1}^{\infty} \frac{(nx)^k}{k!\mathbb{B}(n+1,k)} \int_0^{\infty} (f)(t) \frac{t^{k-1}}{(1+t)^{k+n+1}} + e^{-nx}(f)(0)$$

Be the Szasz-Mirakjan-Beta operators where $f \in \boldsymbol{C_y}[0,\infty)$ such that $|f(t)| \le M(1+t)^y$ for some $M > 0, y > 0$, $\mathbb{B}(n+1,k) = \int_0^{\infty} \frac{t^{k-1}}{(1+t)^{k+n+1}} dt$

In this paper, we will study the approximation of unbounded function in $L_{p,\mathcal{C}_n}(\mathbb{X})$-spaces by using Szasz-Mirkjan-Beta operators as new approach using **Korevkin Theorem** which states the following.

[Korevkin Theorem] [3]

Let $\mathbb{L}_n$ be a linear positive monotone operator such that

1) $\mathbb{L}_n(1,x) = 1$
2) $\mathbb{L}_n(t,x) = x + \alpha(x)$
3) $\mathbb{L}_n(t^2,x) = x^2 + B(x)$

Then for any $f \in C[a,b]$

$$\|\mathbb{L}_n(f,.) - f(.)\|_P \le 3\omega_k\left(f, \sqrt{(B(x) - 2x\alpha(x))}\right)_P \qquad (1.2)$$

2. Preliminary

The following mathematical concepts are needed:

Definition 2.1: [3]

Let $f \in L_p(\mathbb{X}), \mathbb{X} = [0,\infty)$ let

$\Delta_h^k(f,x) = \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} f\left(x - \frac{kh}{2} + ih\right) : x \mp \frac{kh}{2} \in \mathbb{X}$ be the

Difference of $k-$ order of f and

$\omega_k(f,\mathcal{S})_p = Sup_{|h|<\mathcal{S}}\left\|\Delta_h^k(f,.)\right\|_p$ is called the usual modulus of Smoothness of f.

Also

$\tau_k(f,\mathcal{S})_p = \|\omega_k(f,\mathcal{S})\|_p$ is the averaged modulus of smoothness of order k of f

Definition 2.2

Let $f \in L_{p,\mathcal{C}_n}(\mathbb{X}), \mathbb{X} = [0,\infty)$ let

$\omega_k^{\star}(f;\mathcal{S})_{p,\mathcal{C}_n} = Sup_{|h|<\mathcal{S}}\left\|\Delta_h^k(f,x)\right\|_{p,\mathcal{C}_n}, \mathcal{S} \geq 0,$

is called $\mathrm{k}-$ order modulus of smoothness of f

Definition 2.3

For $f \in L_{p,\mathcal{C}_n}(\mathbb{X}), \mathbb{X} = [0,\infty)$.

Let introduce a new sequence of linear operator

$R_n^*(f,x)$ of Szasz-Mirakjan-Beta type operators to approximate a function $f(x)$ belongs to the space $L_{p,\mathcal{C}_n}(\mathbb{X}), \mathbb{X} = [0,\infty)$, $n,k \in N$ as follows:

$$R_n^*(f,x) = e^{-nx}(f\mathcal{C}_n)(0) + e^{-nx}\sum_{k=i}^{\infty}\frac{(nx)^k}{k!\mathbb{B}(n+1,k)}\int_0^{\infty}(f\mathcal{C}_n)(t)\frac{t^{k-1}}{(1+t)^{k+n+1}}$$

Where $x \in [0,\infty), \mathbb{B}(n+1,k) = \int_0^{\infty}\frac{t^{k-1}}{(1+t)^{k+n+1}}dt$

Remark 2.3.1

For every $n \in N$ we have

$$e^{-nx}\sum_{k=1}^{\infty}\frac{(nx)^k}{k!} = e^{-nx}\left[\frac{nx}{1}+\frac{(nx)^2}{2}+\frac{(nx)^3}{6}+\cdots\right] = e^{-nx}.e^{nx} = 1 \qquad (2.1)$$

Remark 2.3.2 [5]:

Properties of Beta function:

a) $\mathbb{B}(n,k) = \mathbb{B}(k,n), n$ and k are are positive integers

b) $\mathbb{B}(n,k) = \frac{(n-1)!}{k(k+1)(k+2)\ldots(k+n-1)}$ if $n \neq k$

c) $\frac{1}{b^{n+1}}\mathbb{B}(n+1,k) = \int_0^{\infty}\frac{t^{k-1}}{(b+t)^{n+k+1}}dt \quad b \in (0,\infty)$

d) $\frac{1}{b^n}\mathbb{B}(k+1,n) = \int_0^{\infty}\frac{t^k}{(b+t)^{n+k+1}}dt$

e) $\frac{1}{b^{n-1}}\mathbb{B}(k+2,n-1) = \int_0^{\infty}\frac{t^{k+1}}{(b+t)^{n+k+1}}dt$

Remark 2.3.3

For n,k are positive integers then

1) $\mathbb{B}(k+1,n) = \frac{k}{n}\mathbb{B}(n+1,k)$

2) $\mathbb{B}(k+2,n-1) = \frac{k(k+1)}{n(n-1)}\mathbb{B}(n+1,k)$

Proof

1) $\mathbb{B}(k+1,n)=\frac{(k+1-1)!}{n(n+1)(n+2)\dots(n+k+1-1)}=\frac{k!}{n(n+1)(n+2)\dots(n+k)}=\frac{k(k-1)!}{n(n+1)(n+2)\dots(n++k-1)}$

$=\frac{k}{n}\mathbb{B}(k,n+1)=\frac{k}{n}\mathbb{B}(n+1,k)$

2) $\mathbb{B}(k+2,n-1)=\frac{(k+2-1)!}{(n-1)(n)(n+1)(n+2)\dots(n-1+k+2-1)}$

$=\frac{(k+1)!}{(n-1)(n)(n+1)\dots(n+k)}=\frac{(k+1)k(k-1)!}{(n-1)(n)(n+1)\dots(n+k)}$

$=\frac{k(k+1)}{n(n-1)}.\frac{(k-1)!}{(n+1)(n+2)\dots(n+k)}=\frac{k(k+1)}{n(n-1)}.\mathbb{B}(k,n+1)=\frac{k(k+1)}{n(n-1)}\mathbb{B}(n+1,k)$ ■

Remark 2.3.4 [4]:

From the linear operator $R_n(f,x)$ let $q_n^*(x)$ be a sequence of real valued continuous functions defined on $[0,\infty)$ with $0\le q_n^*(x)<\infty$ then let define the following positive linear operator.

$$M_n^*(f,x)=a^{n+1}\left(e^{-nq_n^*(x)}\sum_{k=1}^{\infty}\frac{\left(nq_n^*(x)\right)^k}{k!B(n+1,k)}\int_0^{\infty}f(t)\frac{t^{k-1}}{(a+t)^{n+k+1}}dt\right)+e^{-nq_n^*(x)}(f(0)\ .$$

Where

$$q_n^*(x)=\frac{1}{n}\left(-1+\sqrt{1+n\left(n-\frac{2}{3}\right)x^2}\right),\quad \forall x\ge 0, n\in\mathbb{N}, a\in[0,\infty)$$

Remark 2.3.5

From the linear positive operator $R_n^*(f,x)$and in order to obtain an approximation process in space of unbounded functions, let's introduce the new linear positive operator as

$$G_n^\star(f,x)=$$

$$\left\{b^{n+1}\left(e^{-ng_n^\star(x)}\sum_{k=1}^{\infty}\frac{\left(ng_n^\star(x)\right)^k}{k!\mathbb{B}(n+1,k)}\int_0^{\infty}(f\mathcal{C}_n)(t)\frac{t^{k-1}}{(b+t)^{n+k+1}}dt\right)+e^{-ng_n^\star(x)}(f\mathcal{C}_n)(0)\right\}$$

Where $g_n^\star(x)$ be a sequence of real valued continuous functions defined on $[0,\infty)$,

$$g_n^\star(x)=\sqrt{\frac{1+n^2x^2-\frac{2}{3}nx^2}{n}}-\frac{1}{n}<\infty\quad ,\forall x\in[0,\infty), n\in\mathbb{N}, b\in[0,\infty)$$

Remark 2.3.6

For $b=1$ and $g_n^\star(x)=x$ we get

$R_n^*(f,x)=G_n^\star(f,x)$

Definition 2.4

For $f\in L_{p,\mathcal{C}_n}(\mathbb{X}), \mathbb{X}=[0,\infty)$ then

$\tau_k^\star(f,\mathcal{S})_{p,\mathcal{C}_n}=\|\omega_k^\star(f,\mathcal{S})\|_{p,\mathcal{C}_n}$ is the averaged modulus of smoothness of order k of f.

3. Auxiliary Results

Here we will prove some results that are useful to prove our main results.

Lemma 3.1

For $f \in L_{p,\mathcal{C}_n}(\mathbb{X}), \mathbb{X} = [0,\infty)$, let $e_i(x) = x^i, i = 0,1,2$

Then for each $n > 1$ we have

1) $R_n^*(e_0, x) = 1$

2) $R_n^*(e_1, x) = x$

3) $R_n^*(e_2, x) = \frac{nx^2}{n-1} + \frac{2n}{n-1}$

Proof

1) Since

$$R_n^*(f, x) = e^{-nx}(f\mathcal{C}_n)_{(0)} + e^{-nx}\sum_{k=1}^{\infty}\frac{(nx)^k}{k!\mathbb{B}(n+1,k)}\int_0^{\infty}(f\mathcal{C}_n)(t)\frac{t^{k-1}}{(1+t)^{k+n-1}}dt.$$

Then

$$R_n^*(e_0, x) = \left(e^{-nx}\sum_{k=1}^{\infty}\frac{(nx)^k}{k!\mathbb{B}(n+1,k)}\int_0^{\infty}\frac{t^{k-1}}{(1+t)^{n+k+1}}dt\right)$$

$$= \left(e^{-nx}\sum_{k=1}^{\infty}\frac{(nx)^k}{k!\mathbb{B}(n+1,k)}.\mathbb{B}(n+1,k)\right) = e^{-nx}\sum_{k=1}^{\infty}\frac{(nx)^k}{k!} = 1 \quad \text{by (2.1)}$$

2) $R_n^*(e_1, x) = e^{-nx}\sum_{k=1}^{\infty}\frac{(nx)^k}{k!\mathbb{B}(n+1,k)}\int_0^{\infty}t\frac{t^{k-1}}{(1+t)^{n+k+1}}dt$

$$= e^{-nx}\sum_{k=1}^{\infty}\frac{(nx)^k}{k!\mathbb{B}(n+1,k)}\int_0^{\infty}\frac{t^k}{(1+t)^{n+k+1}}dt$$

$$= \left(e^{-nx}\sum_{k=1}^{\infty}\frac{(nx)^k}{k!\mathbb{B}(n+1,k)}.\mathbb{B}(k+1,n)\right) \text{ by remark } (2.3.2{:}\, d) \text{ for } b = 1$$

$$= \left(e^{-nx}\sum_{k=2}^{\infty}\frac{nx(nx)^{k-1}}{k(k-1)!\mathbb{B}(n+1,k)}.\frac{k}{n}\mathbb{B}(n+1,k)\right) \quad \text{by remark } (2.3.3{:}\, 1)$$

$= \left(e^{-nx}\sum_{k=2}^{\infty}\frac{x(nx)^{k-1}}{(k-1)!}\right)$ let $k - 1 = j$ then by (2.1) we get

$$R_n^*(e_1, x) = e^{-nx}\sum_{j=1}^{\infty}\frac{x(nx)^j}{j!} = x\left(e^{-nx}\sum_{j=1}^{\infty}\frac{(nx)^j}{j!}\right) = x.1 = x$$

3) $R_n^*(e_2, x) = \left(e^{-nx}\sum_{k=1}^{\infty}\frac{(nx)^k}{k!\mathbb{B}(n+1,k)}\int_0^{\infty}t^2\frac{t^{k-1}}{(1+t)^{n+k+1}}dt\right)$ where $(f\mathcal{C}_n)_{(t)} = e_2 = t^2$

$$= e^{-nx}\sum_{k=1}^{\infty}\frac{(nx)^k}{k!\mathbb{B}(n+1,k)}\int_0^{\infty}\frac{t^{k+1}}{(1+t)^{n+k+1}}dt$$

$$= \left(e^{-nx}\sum_{k=1}^{\infty}\frac{(nx)^k}{k!\mathbb{B}(n+1,k)}.\mathbb{B}(k+2,n-1)\right) \text{ by remark } (2.3.2{:}\, \mathrm{e}) \text{for } b = 1$$

$$= \left(e^{-nx}\sum_{k=1}^{\infty}\frac{(nx)^k}{k!\mathbb{B}(n+1,k)}.\frac{k(k+1)}{n(n-1)}\mathbb{B}(n+1,k)\right) \quad \text{by remark } (2.3.3{:}\, 2)$$

$$= e^{-nx}\sum_{k=1}^{\infty}\frac{(nx)^k}{k!}.\frac{k(k+1)}{n(n-1)} = e^{-nx}\sum_{k=2}^{\infty}\frac{(nx)(nx)^{k-1}}{k(k-1)!}.\frac{k(k+1)}{n(n-1)}$$

$= \left(e^{-nx}\sum_{k=2}^{\infty}\frac{x(nx)^{k-1}}{(k-1)!}.\frac{(k+1)}{(n-1)}\right)$ let $k - 1 = j$ **then**

$$R_n^*(e_2, x) = e^{-nx}\sum_{j=1}^{\infty}\frac{x(nx)^j}{j!}.\frac{(j+2)}{n-1}$$

$$= e^{-nx}\sum_{j=1}^{\infty}\frac{(nx)^j}{j!}.\frac{xj}{n-1} + e^{-nx}\sum_{j=1}^{\infty}\frac{(nx)^j}{j!}.\frac{2x}{n-1}$$

$$= \left(e^{-nx}\sum_{j=2}^{\infty}\frac{(nx)(nx)^{j-1}}{j(j-1)!}.\frac{xj}{n-1} + 1.\frac{2x}{n-1}\right) \quad \text{let } j-1=k$$

$$R_n^*(e_2,x) = e^{-nx}\sum_{k=1}^{\infty}\frac{(nx)(nx)^k}{(k+1)k!}.\frac{x(k+1)}{n-1} + \frac{2x}{n-1} \quad \text{and by (2.1) we get}$$

$$= e^{-nx}\sum_{k=1}^{\infty}\frac{(nx)^k}{k!}.\frac{nx^2}{n-1} + \frac{2x}{n-1} = 1.\frac{nx^2}{n-1} + \frac{2x}{n-1} = \frac{nx^2}{n-1} + \frac{2x}{n-1}$$

So $R_n^*(e_2,x) = \frac{nx^2}{n-1} + \frac{2x}{n-1}$ ■

Lemma 3.2

For $f \in L_{P,\mathcal{C}_n}(\mathbb{X})$ with $f(x) = 0$ for $x > a > 0, \mathbb{X} = [0,a)$ then

$$\omega_k^\star(f;\mathcal{S})_{p,\mathcal{C}_n} \le \tau_k^\star(f,\mathcal{S})_{p,\mathcal{C}_n} \le a^{\frac{1}{P}}\omega_k^\star(f,\mathcal{S})_{p,\mathcal{C}_n}$$

Proof

$$\tau_k^\star(f,\mathcal{S})_{p,\mathcal{C}_n} = \left\|\omega_k^\star(f,\mathcal{S})_{p,\mathcal{C}_n}\right\|_{p,\mathcal{C}_n} = \int_{\mathbb{X}} \left(\left(\left|\omega_k^\star(f,\mathcal{S})_{p,\mathcal{C}_n}\right|\right)^p dx\right)^{\frac{1}{p}}$$

$$= \left\{\int_{\mathbb{X}} \left[\!\left[Sup_{|h|<\mathcal{S}} \left[\int_0^a \left|\Delta_h^k(f\mathcal{C}_n)_{(t)}\right|^P dt : t, t+kh \in \left[x - \frac{k\mathcal{S}}{2}, x + \frac{k\mathcal{S}}{2}\right] \cap [0,a]\right]\right]\!\right] dx\right\}^{\frac{1}{P}}$$

Applying holder's inequality, we obtain

$$\tau_k^\star(f,\mathcal{S})_{p,\mathcal{C}_n} \le \left\{\int_{\mathbb{X}} \left(Sup_{|h|<\mathcal{S}}\left\{\left|\Delta_h^k(f\mathcal{C}_n)_{(t)}\right|\right\}\right)^p dx\right\}^{\frac{1}{p}} \left\{\int_0^a dx\right\}^{\frac{1}{p}}$$

$$\le \omega_k^\star(f,\mathcal{S})_{p,\mathcal{C}_n}.(a-0)^{\frac{1}{p}} = a^{\frac{1}{p}}\omega_k^\star(f,\mathcal{S})_{p,\mathcal{C}_n}$$

then

$$\tau_k^\star(f,\mathcal{S})_{p,\mathcal{C}_n} \le a^{\frac{1}{p}}\omega_k^\star(f,\mathcal{S})_{p,\mathcal{C}_n} \quad \text{new}$$

$$\omega_k^\star(f,\mathcal{S})_{p,\mathcal{C}_n} = \underset{|h|<\mathcal{S}}{Sup}\left\|\Delta_h^k(f\mathcal{C}_n,x)\right\|_{p,\mathcal{C}_n}$$

$$= \underset{|h|<\mathcal{S}}{Sup}\left\{\int_{\mathbb{X}} \left(\left\{\left|\Delta_h^k(f\mathcal{C}_n)(t)\right|\right\}\right)^p dx\right\}^{\frac{1}{p}} \le \left\{\int_0^a \left(\left|\Delta_h^k(f\mathcal{C}_n,u)\right|\right)^p dx\right\}^{\frac{1}{p}}$$

$$\le \left\{\int_0^a \left(\omega_k^\star(f\mathcal{C}_n,\mathcal{S})\right)^P dx\right\}^{\frac{1}{P}} \le \left\{\int_{\frac{kh}{2}}^{a-\frac{kh}{2}} \left(\omega_k^\star(f\mathcal{C}_n,\mathcal{S})\right)^P dx\right\}^{\frac{1}{P}} = \tau_k^\star(f;\mathcal{S})_{p,\mathcal{C}_n}$$

then $\omega_k^\star(f,\mathcal{S})_{p,\mathcal{C}_n} \le \tau_k^\star(f,\mathcal{S})_{p,\mathcal{C}_n} \le a^{\frac{1}{P}}\omega_k^\star(f,\mathcal{S})_{p,\mathcal{C}_n}$ ■(3.1)

Lemma 3.3

For $f \in L_{P,\mathcal{C}_n}(\mathbb{X})$ with $f(x) = 0$ for $x > a > 0, \mathbb{X} = [0,a)$ we have

$$\tau_k^\star(f,\mathcal{S})_{p,\mathcal{C}_n} \le \tau_{k-1}^\star\left(f', \frac{k}{k-1}\mathcal{S}\right)_{p,\mathcal{C}_n}, \quad k > 1$$

Proof

To prove this lemma we shall use the identity by **[3]**

$\left[\Delta_h^k(f)_{(t)} = \int_0^h \Delta_h^{k-1} f'_{(t+u)}\, du,\ h > 0 \quad , f' = (f\mathcal{C}_n)' \ \right]$. Then

$$\tau_k^\star(f,\mathcal{S})_{p,\mathcal{C}_n} = \|\omega_k^\star(f,\mathcal{S})\|_{p,\mathcal{C}_n} = \left\{\int_0^a |\omega_k^\star(f,\mathcal{S})|^P\, dx\right\}^{\frac{1}{P}}$$

$$\leq \left\{\int_0^a \left| Sup_{|h|<\mathcal{S}} \left\{\int_0^\infty \left|\Delta_h^k(f\mathcal{C}_n)_{(t)}\right|^P dt\right\}^{\frac{1}{P}} \right|^P dx\right\}^{\frac{1}{P}}$$

$$\leq \left\{\int_0^\infty \left| Sup_{|h|<\mathcal{S}} \left\{\int_0^\infty \left|\int_0^h \Delta_h^{k-1}(f\mathcal{C}_n)'_{(t+u)} du\right|^P dt\right\}^{\frac{1}{P}} \right|^P dx\right\}^{\frac{1}{P}}$$

Applying holder's inequality, we obtain

$$\tau_k^\star(f,\mathcal{S})_{p,\mathcal{C}_n} \leq \left\{\int_0^\infty \left| Sup_{|h|<\mathcal{S}} \left\{\int_0^\infty \left|\Delta_h^{k-1}(f\mathcal{C}_n)'_{(t+u)} du.\ \right|^P\right\}^{\frac{1}{P}} \int_0^h dx \right|^P dx\right\}^{\frac{1}{P}}$$

$$\leq \underset{|h|<\mathcal{S}}{Sup} \left\{\int_0^\infty \{|\omega_{k-1}^\star((f\mathcal{C}_n)',\mathcal{S}^*)|^P . h\} dx\right\}^{\frac{1}{P}}$$

$\leq Sup\ \tau_{k-1}^\star(f',\mathcal{S}^*)_{p,\mathcal{C}_n} . h \leq \mathcal{S}\tau_{k-1}^\star(f',\mathcal{S}^*)_{p,\mathcal{C}_n}$, Let $\mathcal{S}^* = \frac{k}{k-1}\mathcal{S}$ then

$$\tau_k^\star(f,\mathcal{S}^*)_{p,\mathcal{C}_n} \leq \mathcal{S}\tau_{k-1}^\star\left(f', \frac{k}{k-1}\mathcal{S}\right)_{p,\mathcal{C}_n} \ \blacksquare$$

Lemma 3.4

For $f \in L_{p,\mathcal{C}_n}(\mathbb{X}), \mu > 0$ we have

$$\tau_k^\star(f,\mu\mathcal{S})_{p,\mathcal{C}_n} \leq \big(2(\mu+1)\big)^{k+1} \tau_k^\star(f,\mathcal{S})_{p,\mathcal{C}_n}$$

Proof

To prove this lemma we must prove that

$\tau_k^\star(f,n\mathcal{S})_{p,\mathcal{C}_n} \leq n^k \tau_k^\star(f,\mathcal{S})_{p,\mathcal{C}_n}$, where n is a positive natural number

Let's use the identity by [**3**]

$\left[\Delta_{nh}^k f(t) = \sum_{i=0}^{(n-1)k} A_i^{n,k} \Delta_h^k f(t+ih)\right]$ where $\sum_{i=0}^{(n+1)k} A_i^{n,k} t^i = (1 + t + t^2 + \cdots + t^{n-1})^k$.

And for $A_i^{n,k} > 0$ for all $i = 0,1,2, \ldots (n-1)k$ we get $\sum_{i=0}^{(n-1)k} A_i^{n,k} = k$

Applying this identity, we obtain

$\left|\Delta_{2nh}^k f(t)\right| \leq \sum_{i=0}^{(2n-1)k} A_i^{2n,k} \Delta_h^k f(t+ih), t, t+2nkh \in \left[x - \frac{kn\mathcal{S}}{2}, x + \frac{kn\mathcal{S}}{2}\right]$

$\leq \sum_{i=0}^{(2n-1)k} A_i^{2n,k} \sum_{j=0}^{2n-1} \left|\Delta_h^k f\left(x - (n-j)\frac{k\mathcal{S}}{2}\right)\right|$. Where

$$t+ih, t+ih+kh \in \left(x - \frac{kn\mathcal{S}}{2} + (j-1)\frac{k\mathcal{S}}{2}, x - \frac{kn\mathcal{S}}{2} + (j+1)\frac{k\mathcal{S}}{2}\right)$$

$j = 1,2, \ldots, 2n-1$

Then

$\left|\Delta_{2nh}^{k} f(t)\right| \le \sum_{i=0}^{(2n-1)k} A_i^{2n,k} \sum_{j=0}^{2n-1} \left|\Delta_h^k f\left(x-(n-j)\frac{k\mathcal{S}}{2}\right)\right|$ If we take the integration of both sides, we get

$$\left[\int_0^\infty \left|\Delta_{2nh}^k (f\mathcal{C}_n)(t)\right|^P dt\right]^{\frac{1}{P}} \le \left(\int_0^\infty \left|\sum_{i=0}^{(2n-1)k} A_i^{2n,k} \sum_{j=0}^{2n-1} \left|\Delta_h^k (f\mathcal{C}_n)\left(x-(n-j)\frac{k\delta}{2}\right)\right|^P dx\right|^P\right)^{\frac{1}{P}}$$

Hence

$$\omega_k^\star(f,x,n\mathcal{S})_{p,\mathcal{C}_n} \le \sum_{i=0}^{(2n-1)k} A_i^{2n,k} \sum_{j=1}^{2n-1} \left(\int_0^\infty \left|\Delta_h^k (f\mathcal{C}_n)\left(x-(n-j)\frac{k\mathcal{S}}{2}\right)\right|^P dx\right)^{\frac{1}{P}}$$

Then

$$\omega_k^\star(f,x;n\mathcal{S})_{p,\mathcal{C}_n} \le (2n)^k(2n-1)\omega_k^\star\left(f, x-(1-n)\frac{k\mathcal{S}}{2}, \mathcal{S}\right)_{p,\mathcal{C}_n}$$

Therefore, if we take the$L_{p,\mathcal{C}_n}$-norm for both sided we get

$\tau_k^\star(f,n\mathcal{S})_{p,\mathcal{C}_n} \le (2n)^k(2n-1)\tau_k^\star(f,\mathcal{S})_{p,\mathcal{C}_n}$

From $[\mu] \le \mu$ and $\mu \le [\mu]+1$ where $[\mu]$ be the smallest integer function we get

$\tau_k^\star(f,\mu\mathcal{S})_{p,\mathcal{C}_n} \le \tau_k^\star(f,([\mu]+1)\mathcal{S})_{p,\mathcal{C}_n}$ by Monotonicity

Let $[\mu]+1 = n$ we get

$\tau_k^\star(f;\mu\mathcal{S})_{p,\mathcal{C}_n} \le \tau_k^\star(f,([\mu]+1)\mathcal{S})_{p,\mathcal{C}_n} = \tau_k^\star(f;n\mathcal{S})_{p,\mathcal{C}_n} \le$

$(2n)^k(2n-1)\tau_k^\star(f,\mathcal{S})_{p,\mathcal{C}_n} = \left(2([\mu]+1)^k(2([\mu]+1)-1)\right)\tau_k^\star(f,\mathcal{S})_{p,\mathcal{C}_n}$

$\le (2(\mu+1)^k)\,(2\mu+1)\,\tau_k^\star(f,\mathcal{S})_{p,\mathcal{C}_n} = 2(\mu+1)^{k+1}\,\tau_k^\star(f,\mathcal{S})_{p,\mathcal{C}_n}$

Then $\tau_k^\star(f,\mu\mathcal{S})_{p,\mathcal{C}_n} \le 2(\mu+1)^{k+1}\,\tau_k^\star(f,\mathcal{S})_{p,\mathcal{C}_n}$ ∎

Lemma 3.5

For $\mathbb{L}_n$ is a linear positive operator in the space $L_{p,\mathcal{C}_n}(\mathbb{X}), \mathbb{X} = [0,a], a > 0$ such that

1) $\mathbb{L}_n(1,x) = 1$
2) $\mathbb{L}_n(t,x) = x + \alpha(x)$
3) $\mathbb{L}_n(t^2,x) = x^2 + B(x)$

Then for any $f \in L_{p,\mathcal{C}_n}(\mathbb{X})$

$$\|\mathbb{L}_n(f) - f\|_{p,\mathcal{C}_n} \le 3\tau_k^\star\left(f, \sqrt{(B(x)-2x\alpha(x))}\right)_{p,\mathcal{C}_n}$$

Proof

Since $\mathbb{L}_n$ is a linear positive operator in the space $L_{p,\mathcal{C}_n}(\mathbb{X})$ which satisfies the above conditions for $f \in L_{p,\mathcal{C}_n}(\mathbb{X})$ then by (1.2) we have

$$\|\mathbb{L}_n(f\mathcal{C}_n) - (f\mathcal{C}_n)\|_p \le 3\omega_k\left((f\mathcal{C}_n), \sqrt{(B(x)-2x\alpha(x))}\right)_p$$

Then by (1.1) we have

$$\|\mathbb{L}_n(f) - (f)\|_{p,\mathcal{C}_n} \le 3\omega_k^\star\left((f), \sqrt{(B(x)-2x\alpha(x))}\right)_{p,\mathcal{C}_n}$$

.Applying (3.1), we obtain

$$\|\mathbb{L}_n(f) - f\|_{p,\mathcal{C}_n} \le 3\omega_k^\star\left(f, \sqrt{(B(x)-2x\alpha(x))}\right)_{p,\mathcal{C}_n} \le 3\tau_k^\star\left(f, \sqrt{(B(x)-2x\alpha(x))}\right)_{p,\mathcal{C}_n}$$

Hence, $\|\mathbb{L}_n(f)-f\|_{p,\mathcal{C}_n} \leq 3\tau_k^\star\big(f,\sqrt{(B(x)-2x\alpha(x)}\big)_{p,\mathcal{C}_n}$ ■

Lemma 3.6

Let $e_i(t)=t^i, i=0,1,2,\ \forall x\geq 0$ and $n>1$ we have

1) $G_n^\star(e_0,x)=1$
2) $G_n^\star(e_1,x)=bg_n^\star(x)=\frac{b\sqrt{1+n^2x^2-\frac{2}{3}nx^2}}{n}-\frac{b}{n}$
3) $G_n^\star(e_2,x)=b^2x^2$

Proof

Since

$G_n^\star(f,x)=$

$$\left\{b^{n+1}\left(e^{-ng_n^\star(x)}\sum_{k=1}^{\infty}\frac{\left(ng_n^\star(x)\right)^k}{k!\mathbb{B}(n+1,k)}\int_0^\infty (f\mathcal{C}_n)(t)\frac{t^{k-1}}{(b+t)^{n+k+1}}dt\right)+e^{-ng_n^\star(x)}(f\mathcal{C}_n)(0)\right\}$$ Then

1) $G_n^\star(e_0,x)=b^{n+1}\left(e^{-ng_n^\star(x)}\sum_{k=1}^{\infty}\frac{\left(ng_n^\star(x)\right)^k}{k!\mathbb{B}(n+1,k)}\int_0^\infty \frac{t^{k-1}}{(b+t)^{n+k+1}}dt\right)$

$=b^{n+1}\left(e^{-ng_n^\star(x)}\sum_{k=1}^{\infty}\frac{\left(ng_n^\star(x)\right)^k}{k!\mathbb{B}(n+1,k)}.\frac{1}{b^{n+1}}.\mathbb{B}(n+1,k)\right)$ by remark (2.3.2: c)

$=e^{-ng_n^\star(x)}\sum_{k=1}^{\infty}\frac{\left(ng_n^\star(x)\right)^k}{k!}=1$ by (2.1)

2) $G_n^\star(e_1,x)=b^{n+1}\left(e^{-ng_n^\star(x)}\sum_{k=1}^{\infty}\frac{\left(ng_n^\star(x)\right)^k}{k!\mathbb{B}(n+1,k)}\int_0^\infty t\frac{t^{k-1}}{(b+t)^{n+k+1}}dt\right)$

$=b^{n+1}\left(e^{-ng_n^\star(x)}\sum_{k=1}^{\infty}\frac{\left(ng_n^\star(x)\right)^k}{k!\mathbb{B}(n+1,k)}\int_0^\infty \frac{t^{k}}{(b+t)^{n+k+1}}dt\right)$

$=b^{n+1}\left(e^{-ng_n^\star(x)}\sum_{k=1}^{\infty}\frac{\left(ng_n^\star(x)\right)^k}{k!\mathbb{B}(n+1,k)}.\frac{1}{b^{n}}\mathbb{B}(k+1,n)\right)$ by remark (2.3.2: d)

$=b^{n+1}\left(e^{-ng_n^\star(x)}\sum_{k=2}^{\infty}\frac{ng_n^\star(x)\left(ng_n^\star(x)\right)^{k-1}}{k(k-1)!\mathbb{B}(n+1,k)}.\frac{1}{b^{n}}.\frac{k}{n}\mathbb{B}(n+1,k)\right)$ by remark (2.3.3: 1)

$=bg_n^\star(x)\left(e^{-ng_n^\star(x)}\sum_{k=2}^{\infty}\frac{\left(ng_n^\star(x)\right)^{k-1}}{(k-1)!}\right)$ let $k-1=j$ then by (2.1) we have

$G_n^\star(e_1,x)=bg_n^\star(x)\left(e^{-ng_n^\star(x)}\sum_{j=1}^{\infty}\frac{\left(ng_n^\star(x)\right)^{j}}{j!}\right)=bg_n^\star(x).1=bg_n^\star(x)$

3) $G_n^\star(e_2,x)=b^{n+1}\left(e^{-ng_n^\star(x)}\sum_{k=1}^{\infty}\frac{\left(ng_n^\star(x)\right)^k}{k!\mathbb{B}(n+1,k)}\int_0^\infty t^2\frac{t^{k-1}}{(b+t)^{n+k+1}}dt\right)$

$=b^{n+1}\left(e^{-ng_n^\star(x)}\sum_{k=1}^{\infty}\frac{\left(ng_n^\star(x)\right)^k}{k!\mathbb{B}(n+1,k)}\int_0^\infty \frac{t^{k+1}}{(b+t)^{n+k+1}}dt\right)$

$=b^{n+1}\left(e^{-ng_n^\star(x)}\sum_{k=1}^{\infty}\frac{\left(ng_n^\star(x)\right)^k}{k!\mathbb{B}(n+1,k)}.\frac{1}{b^{n-1}}\mathbb{B}(k+2,n-1)\right)$ by remark (2.3.2: e)

$=b^{n+1}\left(e^{-ng_n^\star(x)}\sum_{k=1}^{\infty}\frac{\left(ng_n^\star(x)\right)^k}{k!\mathbb{B}(n+1,k)}.\frac{1}{b^{n-1}}.\frac{k(k+1)}{n(n-1)}\mathbb{B}(n+1,k)\right)$ by remark (2.3.3: 2)

$$= b^{n+1}\left(e^{-ng_n^\star(x)}\sum_{k=2}^{\infty}\frac{ng_n^\star(x)\left(ng_n^\star(x)\right)^{k-1}}{k(k-1)!}\cdot\frac{1}{b^{n-1}}\cdot\frac{k(k+1)}{n(n-1)}\right)$$

$$= b^2\left(e^{-ng_n^\star(x)}\sum_{k=2}^{\infty}\frac{g_n^\star(x)\left(ng_n^\star(x)\right)^{k-1}}{(k-1)!}\cdot\frac{k+1}{n-1}\right) \text{ let } k-1=j$$

$$= b^2\left(e^{-ng_n^\star(x)}\sum_{j=1}^{\infty}\frac{g_n^\star(x)\left(ng_n^\star(x)\right)^{j}}{j!}\cdot\frac{(j+2)}{n-1}\right)$$

$$= b^2\left(e^{-ng_n^\star(x)}\sum_{j=1}^{\infty}\frac{\left(ng_n^\star(x)\right)^{j}}{j!}\cdot\frac{g_n^\star(x).j}{n-1} + e^{-ng_n^\star(x)}\sum_{j=1}^{\infty}\frac{\left(ng_n^\star(x)\right)^{j}}{j!}\cdot\frac{2g_n^\star(x)}{n-1}\right)$$

$$= b^2\left(e^{-ng_n^\star(x)}\sum_{j=2}^{\infty}\frac{ng_n^\star(x)\left(ng_n^\star(x)\right)^{j-1}}{j(j-1)!}\cdot\frac{g_n^\star(x)j}{n-1} + 1.\frac{2g_n^\star(x)}{n-1}\right) \text{ let } j-1=k$$

$$= b^2\left(e^{-ng_n^\star(x)}\sum_{k=1}^{\infty}\frac{\left(ng_n^\star(x)\right)^{k}}{k!}\cdot\frac{n(g_n^\star)^2(x)}{n-1} + \frac{2g_n^\star(x)}{n-1}\right) \text{ by (2.1) we get}$$

$$= b^2\left(1.\frac{n(g_n^\star)^2(x)}{n-1} + \frac{2g_n^\star(x)}{n-1}\right) = \frac{b^2 n(g_n^\star)^2(x)}{n-1} + \frac{2b^2 g_n^\star(x)}{n-1}$$

then $G_n^\star(e_2,x) = \frac{b^2 n(g_n^\star)^2(x)}{n-1} + \frac{2b^2 g_n^\star(x)}{n-1}$

and for $g_n^\star(x)(x) = \frac{\sqrt{1+n^2x^2-\frac{2}{3}nx^2}-1}{n}$ we have $G_n^\star(e_2,x) = b^2x^2$ ■

4. Main Results

In this section we will get the approximation for $f \in L_{p,\mathcal{C}_n}(\mathbb{X})$ by using $R_n^*(f,x)\ and\ G_n^\star(f,x)$ operators.

Theorem 4.1

For $f \in L_{p,\mathcal{C}_n}(\mathbb{X}), \mathbb{X} = [0,a], a > 0, n > 1$

$\|R_n^*(f,.) - f(.)\|_{p,\mathcal{C}_n} \le C\tau_k^\star\left(f, \frac{1}{\sqrt{n-1}}\right)_{p,\mathcal{C}_n}$ where

$C = 3\left(2\left(\sqrt{a^2+2a}+1\right)^{k+1}\right)$

Proof

For $f \in L_{p,\mathcal{C}_n}(\mathbb{X}), \mathbb{X} = [0,a], a > 0,$ then by lemma (3.1) we have

1) $R_n^*(e_0,x) = R_n^*(1,x) = 1$

2) $R_n^*(e_1,x) = R_n^*(t,x) = x = x + 0 = x + \alpha(x)$ where $\alpha(x) = 0$

3) $R_n^*(e_2,x) = R_n^*(t^2,x) = \dfrac{nx^2+2x}{n-1} = x^2 + \dfrac{nx^2+2x}{n-1} - x^2 = x^2 + B(x)$

where $B(x) = \dfrac{nx^2+2x}{n-1} - x^2$

and since $x \in [0,a), a > 0$ then for $n > 2$ we have

$\dfrac{nx^2+2x}{n-1} > x^2$ we get $\dfrac{nx^2+2x}{n-1} - x^2 > 0$

And since R_n^* be a linear, monotone and positive operators then by using lemma (3.5) we get

$\|R_n^*(f,.)-f(.)\|_{p,\mathcal{C}_n} \leq 3\tau_k^\star\left(f,\sqrt{B(x)-ax\alpha(x)}\right)_{p,\mathcal{C}_n}$

$= 3\tau_k^\star\left(f,\sqrt{\frac{nx^2+2x}{n-1}-x^2}\right)_{p,\mathcal{C}_n} \quad = 3\tau_k^\star\left(f,\sqrt{\frac{nx^2+2x-nx^2+x^2}{n-1}}\right)_{p,\mathcal{C}_n}$

$= 3\tau_k^\star\left(f,\sqrt{\frac{x^2+2x}{n-1}}\right)_{p,\mathcal{C}_n} \quad \leq 3\tau_k^\star\left(f,\sqrt{\frac{a^2+2a}{n-1}}\right)_{p,\mathcal{C}_n}$

then $\|R_n^*(f,.)-f(.)\|_{p,\mathcal{C}_n} \leq 3\tau_k^\star\left(f,\sqrt{\alpha^2+2a}.\frac{1}{\sqrt{n-1}}\right)_{p,\mathcal{C}_n} \leq$

$3\left(2\left(\sqrt{a^2+2a}+1\right)^{k+1}\right)\tau_k^\star\left(f,\frac{1}{\sqrt{n-1}}\right)_{p,\mathcal{C}_n}$ by lemma (3.4)

let $C=3\left(2\left(\sqrt{\alpha^2+2a}+1\right)^{k+1}\right)$ we get

$\|R_n^*(f,.)-f(.)\|_{p,\mathcal{C}_n} \leq C\tau_k^\star\left(f,\frac{1}{\sqrt{n-1}}\right)_{p,\mathcal{C}_n}$ ■

Theorem 4.2

For $f \in L_{p,\mathcal{C}_n}(\mathbb{X}), \mathbb{X}=[0,a]$ with $f(x)=0$ for $x>a>0$ then

$\|R_n^*(f,.)-f(.)\|_{p,\mathcal{C}_n} \leq \frac{C}{\sqrt{n-1}}\tau_{k-1}^\star\left(f',\frac{k}{k-1}\frac{1}{\sqrt{n-1}}\right)_{p,\mathcal{C}_n}$ where

$C=3\left(2\left(\sqrt{\alpha^2+2a}+1\right)^{k+1}\right), k>1$

Proof

For $f \in L_{p,\mathcal{C}_n}(\mathbb{X}), \mathbb{X}=[0,a]$ by theorem (4.1) we get

$\|R_n^*(f,.)-f(.)\|_{p,\mathcal{C}_n} \leq C\tau_k^\star\left(f,\frac{1}{\sqrt{n-1}}\right)_{p,\mathcal{C}_n}$ where $C=3\left(2\left(\sqrt{\alpha^2+2a}+1\right)^{k+1}\right)$

Applying lemma (3.3), we obtain

$\leq \frac{C}{\sqrt{n-1}}\tau_{k-1}^\star\left(f',\frac{1}{\sqrt{n-1}}\right)_{p,\mathcal{C}_n}$ .Then $\|R_n^*(f,.)-f(.)\|_{p,\mathcal{C}_n} \leq \frac{C}{\sqrt{n-1}}\tau_{k-1}^\star\left(f',\frac{1}{\sqrt{n-1}}\right)_{p,\mathcal{C}_n}$ ■

Theorem 4.3

For $f \in L_{p,\mathcal{C}_n}(\mathbb{X}), \mathbb{X}=[0,a),\ a>0$ we have

$\lim_{n\to\infty}\|R_n^*(f,.)-f(.)\|_{p,\mathcal{C}_n}=0$

Proof

By using theorem (4.1) we get

$\|R_n^*(f,.)-f(.)\|_{p,\mathcal{C}_n} \leq C\tau_k^\star\left(f,\frac{1}{\sqrt{n-1}}\right)_{p,\mathcal{C}_n}$ we have

$\lim_{n\to\infty}\|R_n^*(f,.)-f(.)\|_{p,\mathcal{C}_n} \leq \lim_{n\to\infty}\tau_k^\star\left(f,\frac{1}{\sqrt{n-1}}\right)_{p,\mathcal{C}_n}$ and since $\lim_{n\to\infty}\frac{1}{\sqrt{n-1}}=0$ then

$\lim_{n\to\infty}\|R_n^*(f,.)-f(.)\|_{p,\mathcal{C}_n} \leq \lim_{n\to\infty}C\tau_k^\star\left(f,\frac{1}{\sqrt{n-1}}\right)_{p,\mathcal{C}_n} = C\tau_k^\star\left(f,\lim_{n\to\infty}\frac{1}{\sqrt{n-1}}\right)_{p,\mathcal{C}_n}$

$= C\tau_k^\star(f,0)_{p,\mathcal{C}_n}=0$

hence, $\lim_{n\to\infty} \|R_n^*(f,.) - f(.)\|_{p,\mathcal{C}_n} = 0$ therefore
$\lim_{n\to\infty} R_n^*(f,x) = f(x)$ unformly on $[0,a), a > 0$ ■

Theorem 4.4
For $f \in L_{p,\mathcal{C}_n}(\mathbb{X}), \mathbb{X} = [0,a],\ a > 0$ we have
$\|G_n^\star(f,.) - f(.)\|_{p,\mathcal{C}_n} \le 3\tau_k^\star\big(f,\delta_{n,x}\big)_{p,\mathcal{C}_n}$ where $\delta_{n,x} = \sqrt{b^2x^2 - 2bxg_n^\star(x) + x^2}$

Proof
By using lemma (3.6), we have
1) $G_n^\star(e_0,x) = G_n^\star(1,x) = 1$
2) $G_n^\star(e_1,x) = G_n^\star = bg_n^\star(x) = x + bg_n^\star(x) - x = x + \alpha(x)$
   where $\alpha(x) = bg_n^\star(x) - x$
3) $G_n^\star(e_2,x) = G_n^\star(t^2,x) = b^2x^2 = x^2 + b^2x^2 - x^2 = x^2 + B(x)$

where $B(x) = b^2x^2 - x^2$
and since $G_n^\star(f)$ be a linear positive operator then by using lemma (3.5) we get
$\|G_n^\star(f,.) - f(.)\|_{p,\mathcal{C}_n} \le 3\tau_k^\star\left(f,\sqrt{B(x) - 2x\alpha(x)}\right)_{p,\mathcal{C}_n}$
$= 3\tau_k^\star\left(f,\sqrt{b^2x^2 - x^2 - 2xbg_n^\star(x) + 2x^2}\right)_{p,\mathcal{C}_n} = 3\tau_k^\star\left(f,\sqrt{b^2x^2 - 2xbg_n^\star(x) + x^2}\right)_{p,\mathcal{C}_n}$
$= 3\tau_k^\star\big(f;\mathcal{S}_{n,x}\big)_{p,\mathcal{C}_n}$ where $\mathcal{S}_{n,x} = \sqrt{b^2x^2 - 2xbg_n^\star(x) + x^2}$
and for $b \to 1, g_n^\star(x) = x, n \to \infty$ we get
$\mathcal{S}_{n,x} = \sqrt{b^2x^2 - 2xbg_n(x) + x^2} \to 0$ then
$\|G_n^\star(f,.) - f(.)\|_{p,\mathcal{C}_n} \to 0$ Thus $G_n^\star(f,x) \cong f(x)$, this means that $G_n^\star(f,x)$ is the best approximation to $f(x)$ ■

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