Catalan's Constant is Irrational

Valerii Sopin

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email: VvS@myself.com

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Abstract

In mathematics, Catalan's constant G is defined by

$$G = \beta(2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = \frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \frac{1}{9^2} - \cdots ,$$

where β is the Dirichlet beta function.

Catalan's constant has been called arguably the most basic constant whose irrationality and transcendence (though strongly suspected) remain unproven. In this paper we show that G is indeed irrational.

Proof

Keeping in mind the Riemann series theorem (also called the Riemann rearrangement theorem), we have

$$\begin{array}{cccccc|l}
\frac{1}{1^2} & - \frac{1}{3^2} & + \frac{1}{5^2} & - \frac{1}{7^2} & + \frac{1}{9^2} & - \cdots & G \\
 & - \frac{2}{3^2} & + \frac{2}{5^2} & - \frac{2}{7^2} & + \frac{2}{9^2} & - \cdots & 2G - \frac{2}{1^2} \\
 & & + \frac{2}{5^2} & - \frac{2}{7^2} & + \frac{2}{9^2} & - \cdots & 2G - \frac{2}{1^2} + \frac{2}{3^2} \\
 & & & - \frac{2}{7^2} & + \frac{2}{9^2} & - \cdots & 2G - \frac{2}{1^2} + \frac{2}{3^2} - \frac{2}{5^2} \\
 & & & & + \frac{2}{9^2} & - \cdots & 2G - \frac{2}{1^2} + \frac{2}{3^2} - \frac{2}{5^2} + \frac{2}{7^2} \\
 & & & & & \cdots & \cdots \\
\hline
\frac{1}{1} & - \frac{1}{3} & + \frac{1}{5} & - \frac{1}{7} & + \frac{1}{9} & - \ldots &
\end{array}$$

Notice that the Leibniz formula for π states that

$$\frac{\pi}{4} = \beta(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots .$$

Moreover, it is easy to see that $\sum\limits_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$ is conditionally convergent. On the another hand, $\sum\limits_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2}$ is **absolutely convergent and we are able to rearrange the terms as we want.**

Let's assume **the contrary**: G is a rational number $\frac{s}{2^k t}$, where t is **odd**. Hence, we have

$$stG = st \sum_{n=0,\ n\nmid t}^{\infty} \frac{(-1)^n}{(2n+1)^2} + st \sum_{m=0}^{\infty} \frac{(-1)^{mt+\lfloor t/2 \rfloor}}{t^2(2m+1)^2} =$$

$$st \sum_{n=0,\ n\nmid t}^{\infty} \frac{(-1)^n}{(2n+1)^2} + ((-1)^{\lfloor t/2 \rfloor} 2^k G \sum_{m=0}^{\infty} \frac{((-1)^t)^m}{(2m+1)^2}) = st \sum_{n=0,\ n\nmid t}^{\infty} \frac{(-1)^n}{(2n+1)^2} + ((-1)^{\lfloor t/2 \rfloor} 2^k G^2).$$

In other words, we obtain the following quadratic equation for G:

$$G^2 - (-1)^{\lfloor t/2 \rfloor} \frac{st}{2^k} G + (-1)^{\lfloor t/2 \rfloor} \frac{st}{2^k} \sum_{n=0,\ n\nmid t}^{\infty} \frac{(-1)^n}{(2n+1)^2}.$$

The last is equal to

$$G^2 - (-1)^{\lfloor t/2 \rfloor} \frac{st}{2^k} G + (-1)^{\lfloor t/2 \rfloor} t^2 G \sum_{n=0,\ n\nmid t}^{\infty} \frac{(-1)^n}{(2n+1)^2}.$$

Since $G \neq 0$, we have the next equation

$$G = (-1)^{\lfloor t/2 \rfloor} \frac{st}{2^k} - (-1)^{\lfloor t/2 \rfloor} t^2 \sum_{n=0,\ n\nmid t}^{\infty} \frac{(-1)^n}{(2n+1)^2}.$$

Indeed, we have

$$G = (-1)^{\lfloor t/2 \rfloor} \frac{st}{2^k} - (-1)^{\lfloor t/2 \rfloor} t^2 (G + \epsilon),$$
$$G = (-1)^{\lfloor t/2 \rfloor} t^2 G - (-1)^{\lfloor t/2 \rfloor} t^2 (G + \epsilon),$$
$$G = -(-1)^{\lfloor t/2 \rfloor} t^2 \epsilon,$$

where

$$\epsilon = -\sum_{m=0}^{\infty} \frac{(-1)^{mt + \lfloor t/2 \rfloor}}{t^2 (2m+1)^2} = -(-1)^{\lfloor t/2 \rfloor} \frac{G}{t^2}.$$

According to the above, we consider the following quadratic equation for t:

$$G = (-1)^{\lfloor t/2 \rfloor} \frac{st}{2^k} - (-1)^{\lfloor t/2 \rfloor} t^2 (G + \epsilon),$$

$$t^2 - \frac{s}{2^k (G+\epsilon)} t + (-1)^{\lfloor t/2 \rfloor} \frac{G}{(G+\epsilon)} = 0.$$

Since $\frac{s}{2^k(G+\epsilon)} > 0$ due to $t > 1$ (G can not be $\frac{s}{2^k}$ for natural s, k: it goes around with the representation $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2}$), we get

$$t = \frac{s}{2^{k+1}(G+\epsilon)} (1 \pm \sqrt{1 - \frac{4(-1)^{\lfloor t/2 \rfloor} G (G+\epsilon)^2 2^{2k}}{(G+\epsilon) s^2}}) =$$
$$= \frac{s}{2^{k+1}(G+\epsilon)} (1 \pm \sqrt{1 - \frac{(-1)^{\lfloor t/2 \rfloor} G (G+\epsilon) 2^{2k+2}}{s^2}}).$$

Using the Taylor series of $\sqrt{1+x}$, we come to

$$t_+ \cong \frac{s}{2^k (G+\epsilon)} - \frac{(-1)^{\lfloor t/2 \rfloor} G 2^k}{s},\ t_- \cong \frac{(-1)^{\lfloor t/2 \rfloor} G 2^k}{s},$$

where t_- is impossible as $G = \frac{s}{2^k t}$ and $t > 1$.

Substituting $G = \frac{s}{2^k t_+}$, we derive

$$t_+ \cong \frac{s}{2^k (G+\epsilon)} - \frac{(-1)^{\lfloor t/2 \rfloor} G 2^k}{s} = \frac{s}{2^k (G+\epsilon)} - \frac{(-1)^{\lfloor t/2 \rfloor}}{t+} = \frac{t_+ G}{(G+\epsilon)} - \frac{(-1)^{\lfloor t/2 \rfloor}}{t+}.$$

According to the above, we consider the following quadratic equation for t_+:

$$t_+^2 \frac{\epsilon}{(G+\epsilon)} + (-1)^{\lfloor t/2 \rfloor} \cong 0.$$

Substituting $\epsilon = -(-1)^{\lfloor t/2 \rfloor} \frac{G}{t^2}$, we derive

$$\frac{-G}{(G+\epsilon)} + 1 \cong 0.$$

So, on the one hand, ϵ can not be close to 0 with any accuracy (it is $1/t^2$), but, on the other hand, accuracy of $\cong$ in the Taylor expansion is $O(1/t^4)$. Note that $1/(1 \pm x)$ and $\sqrt{1 \pm x}$ are different as series. Hence, the last equation can not be fulfilled. **Q.E.D.**

Remark 1. *There exists the following integration*

$$\int_0^\infty \frac{1}{1+x^2}\cos(kx)dx = \frac{\pi}{2}e^{-k}.$$

One way to see it is via the Fourier inversion theorem: we know that the Fourier transform of a function has a unique inverse. This carries over to the cosine transform as well. Moreover, the unique continuous function on the positive real axis with Fourier transform $\frac{1}{1+x^2}$ *is* e^{-k}.

Notice that if

$$I_n = \int \frac{x^n}{1+x^2}dx,$$

then

$$I_{n+2} + I_n = \frac{x^n}{n+1} + C.$$

Remark 2. *Are all* $\{1, {}^n\pi \mid n \in \mathbb{N}\}$ *linearly independent over* $\mathbb{Q}$*, where* ${}^n x$ *is tetration? Meaning none of exponents is an integer (we have not known that* $\pi^{\pi^{\pi^{\pi}}}$ *(56 digits) is not an integer).*

Moreover, at least one of e^e *and* e^{e^2} *must be transcendental due to W. D. Brownawell.*

Remark 3. *Is* $e+\pi$ *irrational?*

Note that $(x-e)(x-\pi) = x^2 - (e+\pi)x + e\pi$*. So, at least one of the coefficients* $e+\pi$*,* $e\pi$ *must be irrational.*

Remark 4. *Is* $\ln(\pi)$ *irrational?*

There exists such representation

$$\frac{\sin(x)}{x} = \prod_{n=1}^{\infty}(1 - \frac{x^2}{n^2\pi^2}).$$

Let $x = \frac{\pi}{2}$ *and then we have the Wallis product formulae for* $\frac{\pi}{2}$*:*

$$\frac{\pi}{2} = \prod_{n=1}^{\infty}\frac{2n}{2n-1}\frac{2n}{2n+1}.$$

Taking logarithms of this, we come to

$$\ln(\pi) = \ln(2) + \sum_{n=1}^{\infty}(2\ln(2n) - \ln(2n-1) - \ln(2n+1)).$$

Remark 5. *Is the Euler–Mascheroni constant* γ *irrational?*

$$\gamma = \lim_{n\to\infty}\left(\sum_{m=1}^{n}\frac{1}{m} - \log(n)\right).$$

Remark 6. *Is the Khinchin's constan* K_0 *irrational?*

$$K_0 = \prod_{n=1}^{\infty}(1 + \frac{1}{n(n+2)})^{\log_2 n}.$$

References

[1] Ivan Morton Niven, *Numbers: Rational and Irrational*, Mathematical Association of America, Year: 1961.